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# ALGEBRAIC STACKS

Contents



## Section 1. Introduction

This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [LMB00].

## Section 2. Definitions

**Subsection** 2.1. **Algebraic spaces.** FIXME.

**Definition 2.1.1.** An algebraic space is a stack $\mathcal{S}$ over Aff such that

(1) every fibre category is setlike, see Categories, Subsection 3.2,
(2) the diagonal morphism $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable by schemes, see Schemes, Subsection 3.1 and
(3) there exists a stack $\mathcal{X}$ representable by a scheme, see Schemes, Subsection 2.5 and an étale surjective morphism $\mathcal{X} \to \mathcal{S}$, see Schemes, Theorem 3.1.2.

*Remark* 2.1.2. If you try to define some kind of more general algebraic space by requiring only that the diagonal is representable by algebraic spaces, and that there is a surjective etale morphism of an algebraic space onto $\mathcal{S}$, then you actually end up with the same notion. (FIXME: internal references, proofs.)

**Subsection** 2.2. **Morphisms representable by algebraic spaces.** Here is the formal definition. Please also see the informal discussion below.

**Definition 2.2.1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over Aff. We say $f$ is representable by algebraic spaces if for every stack $\mathcal{S}$ representable by a scheme (see Schemes, Definition 2.5.1), and every morphism $\mathcal{U} \to \mathcal{Y}$, the 2-fibre product $\mathcal{S} \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

Informal discussion. Suppose that, with the notation of the definition, $S$ represents $\mathcal{S}$. Suppose that $W$ is a scheme and that $\mathrm{Aff}/W \to \mathcal{S} \times_{\mathcal{Y}} \mathcal{X}$ is etale and surjective.

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According to Schemes, Lemma 2.5.5 we get a morphism of schemes $g : W \to S$ and a 2-commutative diagram of stacks

$$\begin{tikzcd} \mathrm{Aff}/W \arrow[r] \arrow[d, "g"] & \mathcal{S} \times_{\mathcal{X}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathrm{Aff}/S & \mathcal{S} \arrow[l, "j"'] \arrow[r] & \mathcal{X} \end{tikzcd}$$

**Definition 2.2.2.** Let $P$ be a property of morphisms of schemes, that is etale local on the source and such that if the morphism $f : X \to Y$ has property $P$, then so does every base change of $f$. (FIXME: introduce base change.) We say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ representable by algebraic spaces has property $P$ if for every diagram as above the morphism of schemes $g : W \to S$ has property $P$.

FIXME. Explain rationale behind this definition: what else could it be?

*Subsubsection* 2.2.1. *Algebraic stacks.* FIXME.

**Definition 2.2.3.** An algebraic stack is a stack $\mathcal{S}$ over Aff such that

(1) the diagonal morphism $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable by algebraic spaces, see Definition, Theorem 2.2.1 and
(2) there exists a stack $\mathcal{X}$ representable by a scheme, see Schemes, Subsection 2.5 and a smooth surjective morphism $\mathcal{X} \to \mathcal{S}$, see Definition 2.2.2.

To continue reading,

(1) visit the next section: Algebraic stacks desirables, Section 1, or
(2) go back to the table of contents: index.html#contents.

## References

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, 2000.