LIE SUPERALGEBRAS NOTES

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The color scheme for these pseminar notes is a green value of 0.420. The seminar is titled "Lie Superalgebras and Categorifcation", run by Cailan Li and Alvaro Martinez. See the seminar website at here and the syllabus here.

Any mistakes are of course my own and not the speakers'[1]. Things called "postmortem remarks" are made by myself after the fact and can therefore be completely wrong.

Contents

[1] i too am extraordinarily humble

1. 01/25 – Lie Superalgebra Fundamentals (Cailan Li)

1.1. **Basic definitions.** We begin with some basic definitions.

Definition 1.1.1. A "super vector space", or "vector superspace"/"superspace", is a $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$. Given a homogeneous vector $v \in V_i$, let $|v| = i \in \mathbb{Z}_2$ denote the "parity" of the vector.
Given a superspace V, let Π be the parity-reversing functor, namely $\Pi(V)_i = V_{i+1}$ for $i \in \mathbb{Z}_2$.

Cailan and the book use $V_{\bar{0}}$ and $V_{\bar{1}}$, but I will write only V_0 and V_1 for convenience. EDIT: Cailan agrees with me.

Definition 1.1.2. A "Lie superalgebra" is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a $\mathbb{Z}_2$-graded bilinear operation $[\Box, \Box] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all homogeneous x, y, z we have

- (skew-supersymmetry) $[x, y] = -(-1)^{|x||y|}[y, x]$;
- (superJacobi) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

In particular, e.g. the bracket of a even and an odd thing is odd. It is not hard to see that by using skew-supersymmetry one can bring superJacobi into the more symmetric form

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]].$$

Example 1.1.3. If A is an associative superalgebra, then it can be made a Lie superalgebra by setting

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

Now that we have defined superspaces, what are morphisms between them?

Definition 1.1.4. A map $f \colon \mathfrak{g} \longrightarrow \mathfrak{h}$ between Lie superalgebras is a homomorphism if f is even (i.e. degree 0) and

$$f([x, y]) = [f(x), f(y)].$$

Example 1.1.5. If $\mathfrak{g}$ is a Lie superalgebra, then $\operatorname{End}\mathfrak{g}$ an associative superalgebra is moreover a Lie superalgebra by the previous example. The "adjoint representation" of $\mathfrak{g}$ is then

$$\begin{aligned} \mathrm{ad} \colon \mathfrak{g} &\longrightarrow \operatorname{End}\mathfrak{g} \\ x &\longmapsto [x, \Box]. \end{aligned}$$

One can check that this is a legit homomorphism because of the superJacobi identity.

Postmortem remark: I think the 'End' here refers to not strict morphisms of superspaces, since $(\operatorname{End}\mathfrak{g})_1$ should be a thing also. This latter thing means degree 1 maps surely.

Remark: because the bracket is $\mathbb{Z}_2$-graded, the restriction to the even part actually lands as $\mathrm{ad}|_{\mathfrak{g}_0} \colon \mathfrak{g}_0 \longrightarrow \operatorname{End}\mathfrak{g}_1$, i.e. $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module via the adjoint action.

Here is the main character for this seminar, $\mathfrak{gl}(m|n)$. Let $V = V_0 \oplus V_1 \cong \mathbb{C}^{m|n}$ be a super vector space, where $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$. Then

$$\mathfrak{gl}(m|n) := \operatorname{End}\mathbb{C}^{m|n}$$

equipped with the bracket from the previous example. Fixing a basis for $\mathbb{C}^{m|n}$, we get a natural form to write things in, namely $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The even part looks like

$$\mathfrak{gl}(m|n)_0 \ni \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

and the odd part looks like

$$\mathfrak{gl}(m|n)_1 \ni \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Evidently, as Lie algebras,

$$\mathfrak{gl}(m|n)_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n),$$

and

$$\mathfrak{gl}(m|n)_1 \cong (\mathbb{C}^m \otimes \mathbb{C}^{n*}) \oplus (\mathbb{C}^{m*} \otimes \mathbb{C}^n)$$

as $\mathfrak{gl}(m|n)_0$-modules. Note that as a set, $\mathfrak{gl}(m|n) \cong \mathfrak{gl}(m+n)$, but it is equipped with a different bracket.

Given a matrix $g \in \mathfrak{gl}(m|n)$, when written in the standard form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ earlier, define

Definition 1.1.6. The "supertrace" is

$$\mathrm{str}(g) = \mathrm{tra}(A) - \mathrm{tra}(D).$$

Here are some facts.

Fact 1.1.7. (1) $\mathrm{str}([g,h]) = 0$ for all $g, h \in \mathfrak{gl}(m|n)$;
(2) $\mathfrak{sl}(m|n) = \{g \in \mathfrak{gl}(m|n) : \mathrm{str}(g) = 0\}$ is a Lie subsuperalgebra of $\mathfrak{gl}(m|n)$;
(3) $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$.

See the book for proofs.

Another familiar structure is that of bilinear forms on a vector space. There is a super version of this also.

Definition 1.1.8. A bilinear form $\langle \square ; \square \rangle$ on a superspace $V = V_0 \oplus V_1$ is "supersymmetric" if

$$\langle v; w \rangle = (-1)^{|v||w|} \langle w; v \rangle.$$

It is said to be even if $\langle \mathrm{even}, \mathrm{odd} \rangle = 0$.

We will mostly be concerned with basic Lie superalgebras in this seminar:

Definition 1.1.9. $\mathfrak{g}$ is a "basic Lie superalgebra", if it admits a nondegenerate even supersymmetric bilinear form.

In fact, according to Cailan, "80% of the time we will be concerned with $\mathfrak{gl}(m|n)$ and the rest 20% we will be concerned with $\mathfrak{sl}(m|n)$". Such things are (almost) always basic.

Lemma 1.1.10. $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ (except $(m,n) = (1,1)$ and $(2,1)$ for $\mathfrak{sl}$; $\mathfrak{gl}$ is always basic) are basic Lie superalgebras.

Proof. Let $\langle x, y \rangle = \mathrm{str}(xy)$. This works. ∎

1.2. **Structural things.** Now we discuss things like Cartan, roots, and other structural things.

1.2.1. *Cartan.*

Definition 1.2.1. Let $\mathfrak{g}$ be basic. Then a "Cartan subalgebra" is a Cartan subalgebra of $\mathfrak{g}_0$, and the "Weyl group" is the Weyl group of $\mathfrak{g}_0$.

In our main case $\mathfrak{gl}(m|n)$, the even part is $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, i.e. the diagonal matrices. Let us denote

$$I(m|n) := \{\overline{1}, \cdots, \overline{m}, 1, \cdots, n\},$$

endowed with a total order

$$\overline{1} < \cdots < \overline{m} < 0 < 1 < \cdots < n.$$

Then the Cartan can be written

$$\mathfrak{h}(\mathfrak{gl}(m|n)) = \bigoplus_{i \in I(m|n)} \mathbb{C} E_{ii}.$$

Note that

$$\langle E_{ii}; E_{jj} \rangle = \begin{cases} 1 & \overline{1} \le i = j \le \overline{m} \\ -1 & 1 \le i = j \le n \\ 0 & i \ne j \end{cases};$$

these minus signs come up because of the supertrace.

1.2.2. *Roots.* Now that we have a notion of the Cartan, it makes sense to ask for a root decomposition.

Definition 1.2.2. Let $\mathfrak{h}$ be the Cartan of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : [h, g] = \alpha(h)g \ \ \forall\, h \in \mathfrak{h}\}.$$

Then the "root system" for $\mathfrak{g}$ is

$$\Phi = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}^\alpha \neq 0\},$$

and we can define "even/odd roots" to be

$$\Phi_0 = \{\alpha \in \Phi : \mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq 0\},$$
$$\Phi_1 = \{\alpha \in \Phi : \mathfrak{g}^\alpha \cap \mathfrak{g}_1 \neq 0\}.$$

It is not obvious to me a priori that a root should be either even or odd, or indeed either. But thankfully in the basic case we can say something stronger structurally.

Theorem 1.2.3. Let $\mathfrak{g}$ be a basic Lie superalgebra. Then

(1)
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha;$$

(2) $\langle \Box; \Box \rangle|_{\mathfrak{h}}$ is nondegenerate and W-invariant;

(3) $\dim \mathfrak{g}^\alpha = 1$ (this relies on nondegeneracy of the previous entry)

(4) Φ_0, Φ_1 are invariant under the action of W on $\mathfrak{h}^*$. (And therefore so is Φ.)

Note that the third fact tells us that in this basic case Φ_0 and Φ_1 are thankfully disjoint.

Let's say something about roots for $\mathfrak{gl}(m|n)$. In the case of $\mathfrak{gl}(m|n)$, by definition the Cartan subalgebra is contained in the even part. This implies that the superbracket is coincides with the usual Lie bracket if the first entry is in the Cartan, i.e. the adjoint action of the Cartan on $\mathfrak{gl}(m|n)$ is the same as that of the Cartan on $\mathfrak{gl}(m+n)$. So the roots of $\mathfrak{gl}(m|n)$ are the same as the roots of $\mathfrak{gl}(m+n)$, except with the additional information of a partition of the roots into even and odd things. Let's say what this partition is. Let $\delta_i, \varepsilon_j \in \mathfrak{h}^*$ for $i \in [m]$ and $j \in [n]$ be a dual basis to $E_{\bar{i}\bar{i}}$ and E_{jj} under $\langle \Box; \Box \rangle$. Let us also denote $\varepsilon_{\bar{i}} = \delta_i$. Then the even roots are

$$\Phi_0 = \{\varepsilon_i - \varepsilon_j : i \neq j \in I(m|n) \text{ with either } i, j > 0 \text{ or } i, j < 0\},$$

and the odd roots are

$$\Phi_1 = \{\delta_i - \varepsilon_j, \varepsilon_k - \delta_l : i, l \in [m],\ j, k \in [n]\}.$$

All of this is just a complicated way of saying that the parity of a root is the sum of the parities of the two things it is a difference of.

Because $\mathfrak{h} \cong \mathfrak{h}^*$ under $h \longmapsto \langle h; \Box \rangle$, the form $\langle \Box; \Box \rangle$ induces a form[2] $(\Box; \Box)$ which is also nondegenerate on $\mathfrak{h}^*$. One can easily verify that

$$(\delta_i; \delta_j) = \delta_{ij}, \quad (\varepsilon_i; \varepsilon_j) = -\delta_{ij}, \quad (\varepsilon_k, \delta_l) = 0.$$

In the case of superalgebras, roots can exhibit some weird behavior which don't arise in the usual cases. In particular they can have 'zero length', a phenomenon called isotropy.

Definition 1.2.4. A root $\alpha \in \Phi$ is "isotropic" if $(\alpha; \alpha) = 0$. Let $\overline{\Phi}_1$ be the set of isotropic odd roots.

It bears saying that isotropic automatically implies odd because even roots are actual roots of the Lie algebra $\mathfrak{g}_0$, and the Killing form is positive-definite on the $\mathbb{Q}$-span of Φ, so that in particular $(\alpha; \alpha) > 0$. (Maybe this argument needs a little checking; what is the relationship between $(\Box; \Box)$ and the Killing?)

Example 1.2.5. In the case $\mathfrak{gl}(1|1)$, consider the (only) odd root $\delta_1 - \varepsilon_1$. Compute $(\delta_1 - \varepsilon_1; \delta_1 - \varepsilon_1) = (\delta_1; \delta_1) + (\varepsilon_1; \varepsilon_1) = 1 - 1 = 0$. So the odd root has zero length...

[2]Postmortem remark, mostly for myself: So I guess this would be the dagger rather than the star in the way I learned Lie algebras. The key is that the star doesn't really make general sense in the context of superalgebras since the denominator $(\alpha; \alpha)$ might be zero.

Note that this example easily generalizes to show that all odd roots of $\mathfrak{gl}(m|n)$ are isotropic. So the moral here is that drawing the root picture for Lie superalgebras in general is rather dangerous because of these 'invisible' roots. So instead let us draw them as roots of $\mathfrak{gl}(m+n)$ instead.

1.2.3. *Positive/simple roots.* Now we discuss positive roots.

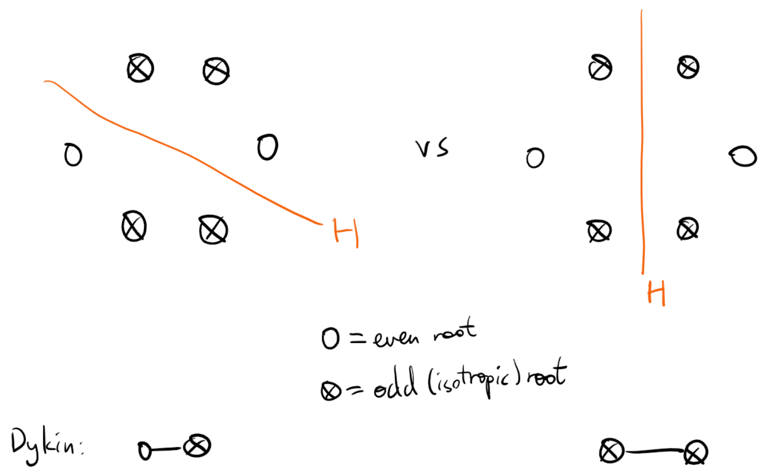
Definition 1.2.6. For $\mathfrak{g} = \mathfrak{gl}(m|n)$ basic, let H be a hyperplane in the picture for $\mathfrak{gl}(m+n)$ not containing any roots and K be the Killing form for $\mathfrak{gl}(m+n)$. Then define

$$\Phi^+(H) = \{\alpha \in \Phi : K(H, \alpha) > 0\}.$$

Let $\Sigma(H)$ be the set of simple roots of $\Phi^+(H)$, i.e. a 'fundamental system'. (Simple here still means not expressible as a positive linear combination of positive roots.)

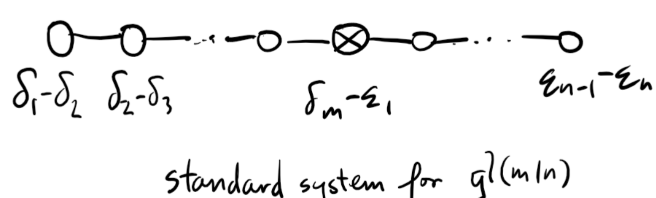
Let me say that Cailan and likely the book use $\Pi(H)$ rather than $\Sigma(H)$. As you might have suspected from the notational choice above, the choice of H actually matters here; different choices may not be conjugate to each other under the Weyl group.

Example 1.2.7. Take $\mathfrak{gl}(2|1)$. There are two odd roots, which are both isotropic, and one even root. Here's a picture:

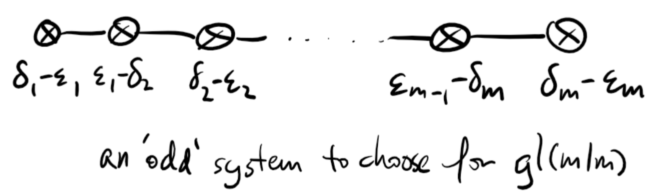


The simple roots, as usual, are the 'closest' ones to the hyperplane H. Note that on the left there is one even simple root and one odd simple root, whereas on the right both simple roots are odd. Hence the Weyl group, being $S_2 \times S_1$ and keeping Φ_i within itself, cannot bring one to the other.

Because of this poor behavior, people tend to stick to a prescribed standard for the simple roots. For $\mathfrak{gl}(m|n)$, this "standard system" is simply consecutive differences of 'diagonal entries'. In terms of Dynkin diagrams this looks like



You can look at nonstandard systems too. If $n = m$, you can have a fundamental system consisting of entirely odd roots (not sure why you would want that); as a picture this is



Now that we have a notion of positive roots, we can say what $\mathfrak{n}^+$ is. Given a choice of hyperplane H, define the nilpotent Lie algebras as follows.

Definition 1.2.8. Given a choice of H, define the "nilpotent" and "Borel" subalgebras as

$$\mathfrak{n}^+(H) = \bigoplus_{\alpha \in \Phi^+(H)} \mathfrak{g}^\alpha,$$
$$\mathfrak{n}^-(H) = \bigoplus_{\alpha \in \Phi^-(H)} \mathfrak{g}^\alpha,$$
$$\mathfrak{b}(H) = \mathfrak{h} \oplus \mathfrak{n}^+(H).$$

Warning: in this definition, it is *not* the case that the Borel is the maximal solvable subalgebra.

1.2.4. *Odd reflections.* Now let us discuss 'odd reflections'. Postmortem remark: As I understand it, this is supposed to atone for the failure of the Weyl group in Example 1.2.7.

Lemma 1.2.9 (Serganova)**.** Let $\mathfrak{g}$ be basic, Σ be a fundamental system for Φ^+, α an isotropic odd root. Then

$$\Phi^+_\alpha := \{-\alpha\} \cup \Phi^+ \setminus \{\alpha\},$$

i.e. Φ^+ except replace α with $-\alpha$, is also a set of positive roots with fundamental system given by

$$\Sigma_\alpha = \{-\alpha\} \cup \{\beta \in \Sigma : (\beta;\alpha) = 0,\ \beta \neq \alpha\} \cup \{\beta + \alpha : \beta \in \Sigma,\ (\beta;\alpha) \neq 0\},$$

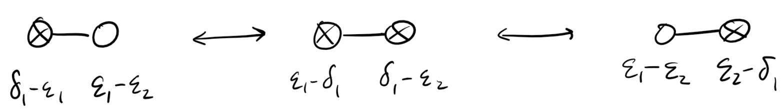
i.e. roughly leave β as is if $(\beta;\alpha) = 0$ and add α to it otherwise.

We can call this procedure r_α, in a satire of the s_α reflection for usual Lie algebras. This r_α is a map of sets, sending

$$r_\alpha \colon \alpha \longmapsto -\alpha,$$
$$\beta \longmapsto \begin{cases} \beta & (\beta;\alpha) = 0 \\ \beta + \alpha & (\beta;\alpha) \neq 0 \end{cases}.$$

Warning: unlike the s_α, this does *not* extend to a linear map.

Example 1.2.10. Consider $\mathfrak{gl}(1|2)$. If we think of this set-theoretically/pictorially as $\mathfrak{gl}(3)$, as we know, there are essentially three choices of hyperplanes in the root picture. Two of these were drawn in Example 1.2.7. The Dynkin pictures for these three choices are



Note well that if we start at the left and pick $\alpha = \delta_1 - \varepsilon_1$, then $\beta = \varepsilon_1 - \varepsilon_2$ has $(\varepsilon_1 - \varepsilon_2; \delta_1 - \varepsilon_1) = -(\varepsilon_1;\varepsilon_1) + (\varepsilon_2;\varepsilon_1) = 1 \neq 0$, so that $r_{\delta_1 - \varepsilon_1}$ tells us to add α to β which brings us to the middle picture. If we start at the middle and pick $\alpha = \delta_1 - \varepsilon_2$, then $(\varepsilon_1 - \delta_1; \delta_1 - \varepsilon_2) = -1 \neq 0$ again and $r_{\delta_1 - \varepsilon_2}$ brings us to the right.

In some sense the Weyl group action can be thought of as 'even reflections'. Then there is a theorem, also due to Serganova:

Theorem 1.2.11. The odd reflections r_α, as defined above, together with the Weyl action, is transitive on the set of fundamental systems.

So once you add in these r_α's you actually hit every possible choice of H.