1) Symplectic variety / manifold

2) Hamiltonian group action

3) Coisotropic preimage

1. Symplectic variety.

X = smooth cplx variety.

A symplectic structure on X is a closed nondegenerate 2-form $\omega \in \Omega^2_{X/\mathbb{C}}$.

Non deg: $\forall x \in X$, ω induces an iso'm b/w $T_x X \simeq T^*_x X$.

Examples

1) $V = \mathbb{C}^{2n} \cong \operatorname{Spec} \mathbb{C}[p_1, p_2, \cdots p_n, q_1, \cdots q_n]$.

($\mathbb{C}^{2n} \cong T^*(\mathbb{C}^n)$; $p_1, \dots, p_n$: "dual coord", $q_1, \dots, q_n$: coord's of $\mathbb{C}^n$.)

$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \cdots + dp_n \wedge dq_n$.

$e_1, \cdots e_n, f_1, \cdots f_n$ be basis of V

$p_1, \cdots p_n, q_1, \cdots q_n$.

$\omega(e_i, e_j) = 0$

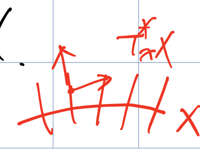
$\omega(f_i, f_j) = 0$

$\omega(e_i, f_j) = \delta_{ij}$.

we say $\{e_1, \cdots e_n, f_1, \cdots f_n\}$ form a Darboux basis.

Fact: any symplectic vector space admits a Darboux basis.

2) X_0 = smooth var / $\mathbb{C}$, $X = T^* X_0$ = cotangent bundle of X.



Then X has a natural symplectic form.

let $(x,\alpha) \in X$, $x \in X_0$, $\alpha \in T_x^* X_0$.

$$\pi: X = T^*X_0 \longrightarrow X_0, \quad (x,\alpha) \longmapsto x \qquad , \quad \pi_* : T_{(x,\alpha)} X \longrightarrow T_x X_0.$$

Define a 1-form λ on X, as follows:

$\forall\ \xi \in T_{(x,\alpha)} X$, let $\langle \lambda, \xi \rangle = \langle \alpha, \pi_* \xi \rangle$

($\alpha \in T_x^* X_0$, $\alpha \to \sum p_i\, dq_i$; $\pi_*\xi \in T_x X_0$, $\pi_*\xi \to \sum c_i \frac{\partial}{\partial q_i}$)

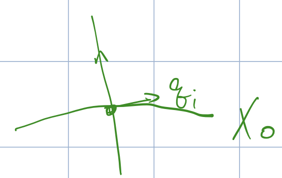
$\omega = d\lambda$, 2-form on X.

To check non-deg, let $q_1, \cdots q_n$ be local coordinates at $x \in X_0$.

$p_1, \cdots p_n$ dual coordinates (linear functions along fibers of T^*X_0) on $X = T^*X_0$ $\to$ local coord's of (x,α) in X.

Then any $\xi \in T_{(x,\alpha)} X$ has the form

$$\xi = \sum b_i \frac{\partial}{\partial p_i} + c_i \frac{\partial}{\partial q_i}$$



$\pi_* \xi = \sum c_i \frac{\partial}{\partial q_i} \in T_x X_0$.

$\lambda = \sum p_i(\alpha)\, dq_i. \implies d\lambda = \sum dp_i \wedge dq_i.$

$\implies$ nondeg. //

3) Coadjoint orbits.

G = Lie group / complex alg. grp. $\mathfrak{g}$ Lie algebra.

$G \curvearrowright \mathfrak{g}$ by Ad, $\rightsquigarrow$ $G \curvearrowright \mathfrak{g}^*$ by Ad^*. (coadjoint action).

Fact: any orbit $\mathbb{O} \subseteq \mathfrak{g}^*$ under Ad^* is symplectic.

$\alpha \in \mathfrak{g}^*$, $\mathbb{O} = G.\alpha$. let $G^\alpha = Stab_G \alpha$.

$\mathbb{O} \cong G/G^\alpha \implies T_\alpha \mathbb{O} \cong \mathfrak{g}/\mathfrak{g}^\alpha$, $\mathfrak{g}^\alpha = Lie(G^\alpha)$.

let $x, y \in \mathfrak{g}$, $\bar{x}, \bar{y}$ image in $\mathfrak{g}/\mathfrak{g}^\alpha$.

$\omega(\bar{x}, \bar{y}) := \langle \alpha, [x,y] \rangle$.

Well-def and non deg $\Leftarrow$ def of $\mathfrak{g}^\alpha$.

Closed $\Leftarrow$ Jacobi identy. on $\mathfrak{g}$. (Prop 1.1.5)

Rmk. If G is semisimple then $\mathfrak{g} \underset{G}{\xrightarrow{\sim}} \mathfrak{g}^*$, so adjoint orbits

= coadjoint orbits

Exercise: • $B \subset SL_2(\mathbb{C})$, $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2 \right\}$.

Then not all adjoint orbits of $B \curvearrowright \mathfrak{b}$ are symplectic.

• compute coadj orbits of B.

4) Special case of T^*X_0. (1.4.9)

G = complex alg grp., $P \subseteq G$ be a closed subgroup (connected).

$\mathfrak{g} \supseteq \mathfrak{p}$ Lie algebras. Example: G = s.s grp, B = Borel

$T^*(G/P)$, symplectic form. $T^*(G/B) \leftarrow$ Import. in Ch 3. coadjoint action

Lemma: $T^*(G/P) \cong G \times_P (\mathfrak{g}/\mathfrak{p})^* =: G \times_P \mathfrak{p}^\perp$.

$\mathfrak{p}^\perp \subseteq \mathfrak{g}^*$ $\quad \mathfrak{p}^\perp = \{ \alpha \in \mathfrak{g}^* \mid \langle \alpha, \mathfrak{p} \rangle = 0 \}$.

$\langle \alpha, \xi \rangle = 0 \ \forall \xi \in \mathfrak{p}$

Pf. $G/P \times \mathfrak{g}$ trivial bundle $/ G/P$.

$G \curvearrowright G/P$, $\rightsquigarrow \mathfrak{g} \to \mathrm{Vect}(G/P)$.

$G/P \times \mathfrak{g} \overset{\varphi}{\twoheadrightarrow} T(G/P)$, $\Leftarrow \forall x \in G/P$, $\forall v \in T_x G/P$, $\exists \xi \in \mathfrak{g}$ st. $\xi . x = v \in T_x G/P$.

$G \supset P$, V is a P-rep, $G \times_P V$, vector bundle on G/P. $[g, v] = [gp, p^{-1}v]$

$G \times_P V = G \times V / \sim$, $\sim : (g, v) = (gp, p^{-1}v)$

$(gP, \xi) \overset{\varphi}{\longmapsto} (gP, \underline{\xi . gP})$; $\gamma(t) \subset G$, $\dot{\gamma}(0) = \xi$;

$\frac{d}{dt}(\gamma(t) g P)|_{t=0}$ $\quad \gamma(t) g P \subseteq gP \Rightarrow g^{-1}\gamma(t) g \subseteq P$.

$\ker \varphi$? $\quad \xi . gP = 0. \Leftrightarrow \xi \in \mathrm{Ad}_g \mathfrak{p}. \Leftarrow \mathrm{Ad}_g^{-1} \xi \in \mathfrak{p}$.

$\parallel$

E over G/P. $\subseteq G/P \times \mathfrak{g}$., $E = \{ (gP, \mathrm{Ad}_g \mathfrak{p}) \}$.

$$E = G \overset{\text{Adjoint}}{\times_P} \mathfrak{p}$$

In fact, $G \times \mathfrak{p} \longrightarrow E$, $(g, \eta) \longmapsto (gP, \mathrm{Ad}_g \eta)$.

$$\begin{array}{ccc} G\times\mathfrak{p} & \longrightarrow & E \\ \downarrow & \nearrow \sim & \\ G\times_P \mathfrak{p} & & \end{array}$$

$(gp, \mathrm{Ad}_p^{-1}\eta) \longmapsto (gpP, \mathrm{Ad}_{gp}\mathrm{Ad}_p^{-1}\eta) = (gP, \mathrm{Ad}_g \eta)$

$$G \overset{\text{triv}}{\times_P} \mathfrak{g} \cong G \overset{\text{Ad}}{\times_P} \mathfrak{g}, \quad (g, \xi) \longmapsto (g, \mathrm{Ad}_g \xi)$$

$$\Rightarrow (G/P \times \mathfrak{g})/E \cong G \overset{\text{Ad}}{\times_P} (\mathfrak{g}/\mathfrak{p}).$$

$(G/P \times \mathfrak{g})/E = T(G/P)$.

Take dual $\Rightarrow$ $T^*(G/P) = G \overset{\text{coad}}{\times_P} (\mathfrak{g}/\mathfrak{p})^* = G \times_P \mathfrak{p}^\perp$. //.

$G \times_P \mathfrak{p}^\perp$; vertical vectors, $\alpha \in \mathfrak{p}^\perp$

vectors induced by G action. $\mathfrak{g} \longrightarrow \mathrm{Vect}(G\times_P \mathfrak{p}^\perp)$

then $\omega(\alpha_1, \alpha_2) = 0$, if $\alpha \in \mathfrak{p}^\perp$.

$\omega(\xi, \eta)|_{(g,\alpha)} = \langle \alpha, \mathrm{Ad}_g [\xi, \eta] \rangle$, if $\xi, \eta \in \mathfrak{g}$

$\omega(\beta, \xi)|_{(g,\alpha)} = \langle \beta, \mathrm{Ad}_g \xi \rangle$, $\beta \in \mathfrak{p}^\perp$, $\xi \in \mathfrak{g}$.

(1.4.11)

Rmk. G/P: quasi projective. realize G/P as an orbit in some $\mathbb{P}(V)$;

G/P need not be affine.

Eg if $P = \text{Borel} \subseteq G$, G/B is projective.

5) Generalization: $\nu \in \mathfrak{g}^*$ s.t. $\nu|_{[\mathfrak{p}, \mathfrak{p}]} = 0$.

then $G \times_P (\nu + \mathfrak{p}^\perp)$ is also a symplectic variety.
"twisted cotangent bundle". (1.4.15)

2. Poisson variety and Hamiltonian action. (§1.2)

Def. A = commutative $\mathbb{C}$-alg, $\{\cdot,\cdot\} : A \otimes_{\mathbb{C}} A \to A$.
is a Poisson bracket if

- $\{\cdot,\cdot\}$ is linear and skew symmetric
- Leibniz rule : $\{fg, h\} = f\{g,h\} + \{f,h\}g$
- Jacobi identity: $\{f\{g,h\}\} + \{g,\{h,f\}\} + \{h\{f,g\}\} = 0$.

A is a Lie alg wrt. $\{\cdot,\cdot\}$.

We say $(A, \{\cdot,\cdot\})$ is a Poisson algebra.

Example, $A = \mathbb{C}[p_1, \cdots p_n, q_1, \cdots q_n]$,
$\{p_i, p_j\} = \{q_i, q_j\} = 0$
$\{p_i, q_j\} = \delta_{ij}$.

If X is a symplectic variety, can define $\{\cdot,\cdot\}$ on $\mathcal{O}_X$.
(If X affine $\Rightarrow \mathcal{O}(X)$ is a Poisson alg;
in general, $\mathcal{O}_X$ is a sheaf of Poisson algs)

For now assume (X, ω) is an affine symp. variety.
Goal: $\{\cdot,\cdot\}$ on $\mathcal{O}(X)$.

let $f, g \in \mathcal{O}(X)$. $df \in \Omega^1_{X/\mathbb{C}} \overset{\omega}{\Longrightarrow} \xi_f \in \mathrm{Vect}(X)$.

i.e. $\omega(-, \xi_f) = df$.

ξ_f the Hamiltonian vect field of f.

Define $\{f, g\} = \omega(\xi_f, \xi_g) = \xi_f g = -\xi_g f$.

- skew symmetric,
- Leibniz rule: $\{f, gh\} = \xi_f(gh)$
- Jacobi identity ? $\Longleftarrow$

A dg argument $\Rightarrow$ $\xi_{\{f,g\}} = [\xi_f, \xi_g]$. (1.2.6)

$\Rightarrow$ $\{f, g\} = \omega(\xi_f, \xi_g)$ is a Poisson bracket on $\mathcal{O}(X)$.

Def. · A vect. field of the form ξ_f, $f \in \mathcal{O}(X)$, is called Hamiltonian.

· A vect. field v is called symplectic if $L_v \omega = 0$.

$\iota_{\xi_f} \omega = -df \Rightarrow v$ is Hamiltonian iff $\iota_v \omega$ is exact.

Cartan magic formula,

$$L_v \omega = d\, \iota_v \omega + \iota_v d\omega \quad (\iota_v d\omega \to 0)$$

Lie derivative ($\swarrow$ L_v)

$L_v\omega = 0 \iff d\,\iota_v\omega = 0$

i.e. v is a symplectic vect field iff $\iota_v \omega$ is closed.

$\iota_v \omega(u) = \omega(v, u)$.

$\Rightarrow$ If $H^1_{dR}(X) = 0$ then all symp v.f. are Hamiltonian. ef

(Perhaps wrong: v vect field, $\rightsquigarrow$ $\Phi(t)$ flow of v. Vect. spaces,

$$\lim_{t\to 0} \frac{\phi^*(t)\omega - \omega}{t} =: L_v\omega. \quad)$$

$$\parallel$$
$$0$$

Exe. $V = \mathbb{C}^{2n}$,

1) $v = \frac{\partial}{\partial p_1}$

$\phi(t)(a_1, \dots a_n, b_1, \dots b_n) = (a_1 + t, a_2, a_3, \dots a_n, b_1, \dots b_n)$

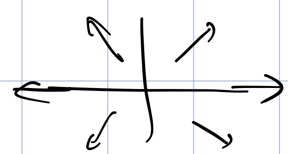
$\phi^*(t)\,\omega = \phi(t)^*(dp_1 \wedge dq_1 + \dots dp_n \wedge dq_n)$

$= \omega.$

$\Rightarrow \lim_{t\to 0} \frac{\phi^*(t)\omega - \omega}{t} = 0 \Rightarrow v$ is symplectic vect field.

2) v is vect field $\sum p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}$

$\phi(t)$ is "rescaling".



$L_v\omega \neq 0$.

(will be a multiple of ω).

Hamiltonian group actions. (1.4)

(X,ω) be symp variety. $G \curvearrowright X$. st $\forall g \in G$,

$g^*\omega = \omega$.

$\Rightarrow \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$, image are symplectic vect. fields.

Q: whether the image are Hamiltonian?

We say $G \curvearrowright (X,\omega)$ is Hamiltonian if $\exists$

a Lie alg hom $\mu^*: \mathfrak{g} \longrightarrow \mathcal{O}(X)$ ($\{\cdot,\cdot\}$ from ω)

$\xi \longmapsto H_\xi$

$$\begin{array}{ccc} & \xi \in \mathfrak{g} & \longrightarrow \ \xi_X \\ \text{s.t.} \ H_\xi \swarrow & \downarrow & \searrow \\ & \mathcal{O}(X) & \longrightarrow \ \mathrm{Vect}_{\mathrm{Symp}}(X) \end{array}$$

commutes

all symplectic vector fields ξ_X are Hamiltonian.

If $H^1_{dR}(X) = 0$ then any symp grp action is Hamiltonian (?)

Can define $\mu : X \longrightarrow \mathfrak{g}^*$, from $\mu^* : \mathfrak{g} \to \mathcal{O}(X)$.

$\circlearrowleft_G$ $\quad \circlearrowleft_{\text{coadj.}}$ $\quad$ $\wr$ extends to $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \longrightarrow \mathcal{O}(X)$

Exe. If G is connected, then μ (μ^*) is G-equivariant.

Examples 1) (V, ω) symp. vect space, $G = Sp(V)$.

then $G \curvearrowright V$ is Hamiltonian

$\forall \xi \in \mathfrak{sp}(V)$, $\underbrace{\mu^*(\xi)}_{\in \mathbb{C}[V]}(v) = \frac{1}{2}\omega(\underbrace{\xi.v}_{\in T_vV \cong V}, v)$

2) If $G \curvearrowright X_0 \Rightarrow G \curvearrowright T^*X_0 =: X$.

Hamiltonian $\quad (x, \alpha) \in T^*X_0$, $x \in X_0$, $\alpha \in T^*_x X_0$

$\underbrace{\mu(x,\alpha)}_{\in \mathfrak{g}^*}(\xi) = \langle \underbrace{\xi.x}_{\in T_xX_0}, \alpha \rangle$, $\quad \xi \in \mathfrak{g}$

Next time: 1.5.1, 1.5.7.