KLR Algebras: Essentials

2/7

0. Some motivation

$\langle [M] : M \in \mathrm{Ob}\,\mathcal{C}_n \rangle / [M \oplus N] = [M] + [N]$

1970s: If $\mathcal{C}_n = \mathbb{C}S_n\text{-mod}$, $K := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} K_0(\mathcal{C}_n)$ has a bialgebra structure, isomorphic to Sym

Features

	Sym $= \varprojlim \mathbb{Z}[x_1, \dots, x_n]^{S_n}$	K
Multiplication:	$p \cdot q$	$[M_1] \cdot [M_2] := [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} M_1 \boxtimes M_2]$
Comultiplication:	$\Delta(x_\ell) = \sum_{m+n=\ell} x_m \otimes x_n$	$\Delta([M]) := [\bigoplus_{n+m=\ell} \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} M]$
Grading:	$\mathbb{Z}_{\geq 0}$ (deg)	$\mathbb{Z}_{\geq 0}$
Distinguished basis:	s_λ	$[S^\lambda]$

Remark: one also has $H^*(BU; \mathbb{Z}) \cong K \cong \mathrm{Sym}$ as Hopf algebras

Question: can we replace Sym with $U_q(\mathfrak{g})$?

First goal of the seminar: $U_q(\mathfrak{g}) = \underbrace{U_q(\mathfrak{n}^-)}_{\text{Categorify this}} \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+)$

(Why this? Rep-theoretic + low dim'l topology reasons see last week's notes)

Cartan datum of $\mathfrak{g}$: $(I, \cdot)$ symmetric, Dynkin diagram Γ (for today's sake): no double edges or loops

Recall that $U_q(\mathfrak{n}^-)$ is $\mathbb{Z}_{\geq 0}[I]$-graded: e.g. $\deg(y_{\alpha_1}^2 y_{\alpha_2} y_{\alpha_4}^3) = 2\alpha_1 + \alpha_2 + 3\alpha_4$

In order to categorify $U_q(\mathfrak{n}^-)$, we need a $\mathbb{Z}[q^{\pm 1}]$-form.

Lusztig (1990): Algebra f : generators θ_i, $i \in I$

relations: $\theta_i \theta_j = \theta_j \theta_i$ if $i \cdot j = 0$ $\longleftrightarrow$ $i \not\!-\, j$

$\theta_i \theta_j \theta_i = \theta_i^{(2)} \theta_j + \theta_j \theta_i^{(2)}$ if $i \cdot j = -1$ $\longleftrightarrow$ $i \text{ — } j$

Integral form: $\mathbb{Z}[q^{\pm 1}]$-algebra generated by $\theta_i^{(a)} := \dfrac{\theta_i^a}{[a]!}$, where $[a]! = [a] \cdot [a-1] \cdots [2]$

$[n] = q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1}$

Theorem: $\theta_i \mapsto y_i$ is an isomorphism $f \longrightarrow U_q(\mathfrak{n}^-)$

For each $\nu \in \mathbb{Z}_{\geq 0}[I]$, we will construct (KLR) algebras $R(\nu)$ so that

$$\bigoplus_{\nu \in \mathbb{Z}_{\geq 0}[I]} K_0(R(\nu)\text{-mod}) \cong f_{\mathbb{Z}[q^{\pm 1}]}$$ as twisted bialgebras

Features

$f_{\mathbb{Z}[q^{\pm1}]}$	$\bigoplus_{\nu\in\mathbb{Z}_{\geq 0}[I]} R(\nu)\text{-pmod}^{fg}$
Element $\theta_{i_1}\cdots\theta_{i_\ell}$	Object $\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_\ell}$
Sum	$\oplus$
Multiplication	$\mathrm{Ind}_{R(\nu)\otimes R(\nu')}^{R(\nu+\nu')}(-\otimes -)$
Comultiplication	$\bigoplus_{\nu+\nu'} \mathrm{Res}_{R(\nu)\otimes R(\nu')}^{R(\nu+\nu')}(-)$
Multiplication by q^a	$(-)\{a\}$
Relations	Isomorphisms
Bilinear form $(,)$	$\mathrm{Hom}^{\bullet}(-,-)$ (graded vector space)
Canonical basis	Projective indecomposables

1. Starting point: Lusztig's algebra f

As in the previous section, $\Gamma = (I, E_\Gamma)$ graph with no double edges or loops.

Notation: for $\nu = \sum \nu_i \cdot i \in \mathbb{Z}_{\geq 0}[I]$, write $|\nu| = \sum \nu_i$

Define
- ${}'f$ = free $\mathbb{Q}(q)$-algebra generated by θ_i, for $i \in I$.
- Comultiplication: $\Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$
- Twisted algebra structure on ${}'f \otimes {}'f$: $(x_1 \otimes x_2)(x_3 \otimes x_4) = q^{-|x_2| \cdot |x_3|} x_1 x_3 \otimes x_2 x_4$
- Bilinear form on ${}'f$: $(1,1) = 1$

$$(\theta_i, \theta_j) = \delta_{ij} \frac{1}{1-q^2}$$

$$(x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2)$$

$$(x_1 x_2, y) = (x_1 \otimes x_2, \Delta(y))$$

- Finally, $f = {}'f / \mathrm{Ker}((\,,))$

Khovanov-Lauda: use this data to define a graphical calculus

2. Construction

$\Gamma = (I, E_\Gamma) \longleftrightarrow$ Cartan pairing: $i \cdot j = \begin{cases} 2 & i = j \\ -1 & i - j \\ 0 & \text{otherwise} \end{cases}$

Fix $\nu = \sum \nu_i \cdot i$. Let $Seq(\nu) = \{$sequences $\vec{i}$ where each $i \in I$ appears ν_i times$\}$

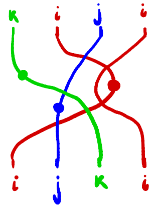
Consider:

- Sequences $\vec{i} \in Seq(\nu)$

  $\nu = 2i + j + k$

  $\vec{i} = k \; i \; j \; i$

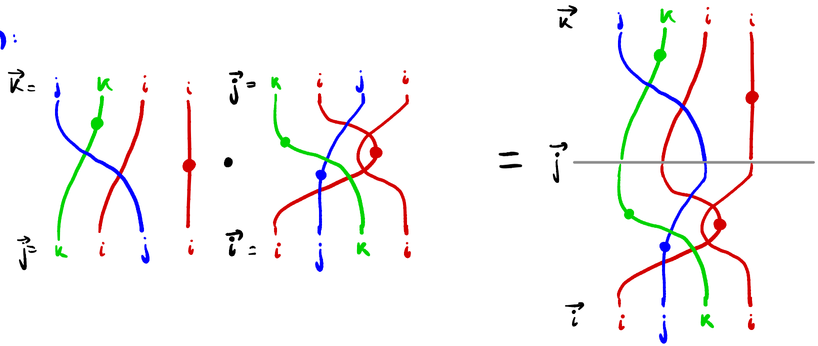
- Diagrams "between them":



We require:

- Diagrams are "braid-like" (no critical points)
- They may carry dots
- Isotopic diagrams are equal (not allowed to create critical points)
- No triple intersections

Multiplication:



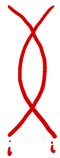
If sequences don't match, set to 0.

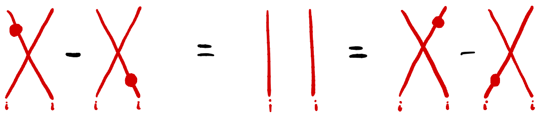
Let ${}_{\vec{j}}R(\nu)_{\vec{i}}$ = $\mathbb{Z}$ - span of diagrams from $\vec{i}$ to $\vec{j}$, subject to local relations
(to be defined on the next slides)

Then $R(\nu) := \bigoplus_{\vec{i},\vec{j}\in \mathrm{Seq}(\nu)} {}_{\vec{j}}R(\nu)_{\vec{i}}$

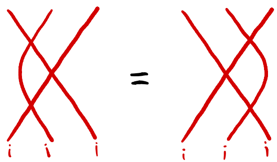
Relations

- One color

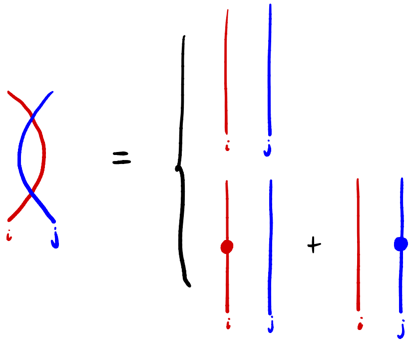
 $= 0$



"NilHecke relations"

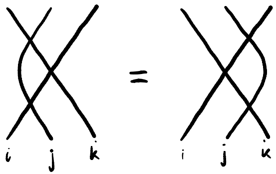


• More than one color

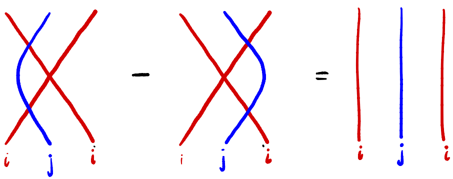


if i —/— j

if i —— j



for all choices of i, j, k UNLESS $i = k$ and i —— j



i —— j

3. Where does this come from?

Recall that $\mathrm{Hom}^\bullet(-,-)$ is supposed to categorify $(\,,\,)$ on $\mathfrak{f}$.

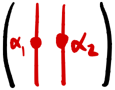
Now $(\theta_i, \theta_i) = \dfrac{1}{1-q^2} = 1 + q^2 + q^4 + \ldots$ so we want a morphism for each degree $0, 2, 4, \ldots$

$\rightsquigarrow$ Define  where deg () = 2 .

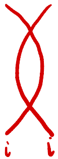
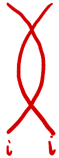
Write  = 

A computation shows:

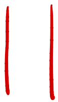
$$(\theta_i^2, \theta_i^2) = (1+q^{-2})\left(\frac{1}{1-q^2}\right)^2 = q^{-1} + 3 + 5q^2 + 7q^4 + 9q^6 + \dots$$

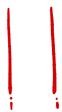
We already have $\deg\Big($  $\Big) = 2\alpha_1 + 2\alpha_2$ This accounts for $\left(\frac{1}{1-q^2}\right)^2 = 1 + 2q^2 + 3q^4 + 4q^6 + \dots$

Need a morphism of degree -2 $\rightsquigarrow$ 

Now since $\deg\Big($  $\Big) = -4$, we must have  $= 0$

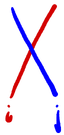
Now the coefficient of q^0 in (θ_i^2, θ_i^2) is 3, so

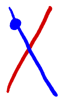
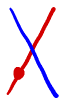
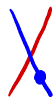
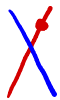
$\mathbb{Z}[q^{\pm 1}] \cdot \{$  ,  ,  ,  ,  $\}$ should have dimension 3

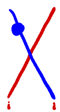
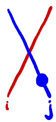
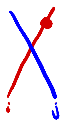
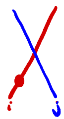
$\rightsquigarrow$  $-$  $=$  $=$  $-$  (see later)

Another computation shows:

If i —— j, $(\theta_i\theta_j, \theta_j\theta_i) = q\left(\frac{1}{1-q^2}\right)^2 = q + 2q^3 + 3q^5 + \dots$

$\rightsquigarrow$  , degree 1

Now $\mathbb{Z}\cdot\{$  ,  ,  ,  $\}$ should have dimension 2

$\rightsquigarrow$  $=$     $=$ 

Remark: Consider a further example:

$(1-q^{2})(\ ,\ )$	$\theta_i\ \ {}_i\ \ {}_j$	$\theta_i\theta_j\theta_i$	$\theta_j\theta_i\theta_i$
$\theta_i\theta_i\theta_j$	$1+q^{-2}$	$q+q^{-1}$	$1+q^{2}$
$\theta_i\theta_j\theta_i$	$q+q^{-1}$	2	$q+q^{-1}$
$\theta_j\theta_i\theta_i$	$1+q^{2}$	$q+q^{-1}$	$1+q^{-2}$

Rmk: $\mathrm{Ker}((\ ,\ )) = \langle (q+q^{-1})\theta_i^2\theta_j - \theta_i\theta_j\theta_i + (q+q^{-1})\theta_j\theta_i^2 \rangle$ Quantum Serre relation

Observation: $(1-q^{-2})\cdot(\theta_{\vec{i}},\theta_{\vec{j}})$ is given by the sum over matchings $\vec{i} \to \vec{j}$

weighted by $q^{-\sum_{\substack{X \\ i_1 i_2}} i_1\cdot i_2}$

Fact: this holds in general $\rightsquigarrow$ Hints at diagrammatic categorification

<u>Remark</u>: This kind of reasoning gets us quite far, though not all the way.

Assume $\Gamma = \bullet$. Then $\mathfrak{f} = U_q^-(\mathfrak{sl}_2)$ acts on $\bigoplus_{k=0}^{N} H^*(Gr(k,N))$ via $\theta \mapsto$ tensoring by $H^*(Fl(k,k+1,N))$

We want to upgrade this to a categorical action, so

- Understand $R(i)$ as a category with object i, morphisms  k
- Then  $\rightsquigarrow$ some 2-morphism $H^*(Fl(k,k+1,N)) \otimes (-) \to H^*(Fl(k,k+1,N)) \otimes (-)$

↳ One possibility: multiplication by the Chern class of the line bundle $\mathbb{C}^{k+1}/\mathbb{C}^k$

$\rightsquigarrow$ Nil Hecke relations

Computations with partial flag varieties justify the other relations.

Alternative motivation: functors on parabolic category $\mathcal{O}$

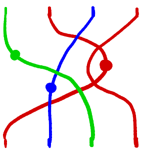
4. Projectives and antiinvolution

Observe that

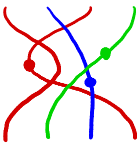
$$P_{\vec{i}} = \bigoplus_{\vec{j} \in \mathrm{Seq}(\nu)} {}_{\vec{j}}R(\nu)_{\vec{i}} \qquad \text{left graded projective}$$

$${}_{\vec{j}}P = \bigoplus_{\vec{i} \in \mathrm{Seq}(\nu)} {}_{\vec{j}}R(\nu)_{\vec{i}} \qquad \text{right graded projective}$$

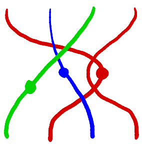
Also, consider



$\xrightarrow[\text{involution}]{\sigma}$ $(-1)^{\# X}$



$\psi \downarrow$ antiinvolution



Remark: $\psi \circ \sigma = \sigma \circ \psi$

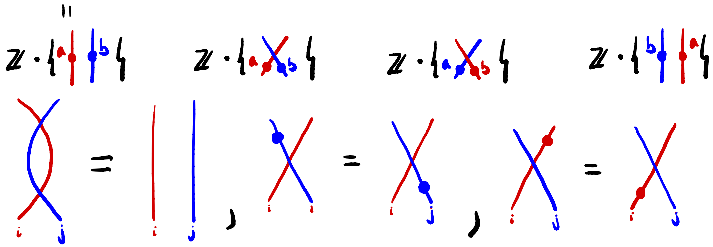
5. First examples

- $\nu = i$ $\qquad \mathrm{Seq}(\nu) = \{ i \}$ $\qquad R(i) = \mathbb{Z}[\bullet]$ deg 2

- $\nu = i + j$, $\quad i \not- j$

  $\mathrm{Seq}(\nu) = \{ ij, ji \}$

  $$R(i+j) = {}_{ij}R(i+j)_{ij} \oplus {}_{ji}R(i+j)_{ij} \oplus {}_{ij}R(i+j)_{ji} \oplus {}_{ji}R(i+j)_{ji}$$

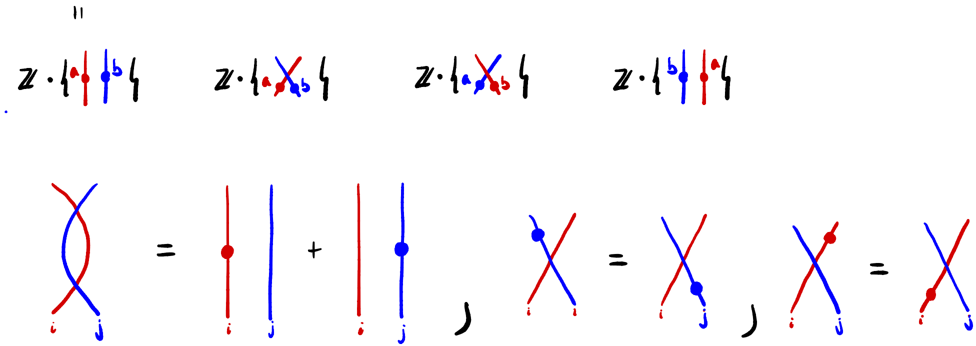


  $$\cong M_2(\mathbb{Z}[x, y])$$

- $\nu = i_1 + \dots + i_m$, $\quad i_k \not- i_l$ for all k, l $\Rightarrow R(\nu) \cong M_{m!}(\mathbb{Z}[x_1, \dots, x_m])$

- $v = i + j \qquad i - j$

$$R(i+j) = {}_{ij}R(i+j)_{ij} \oplus {}_{ji}R(i+j)_{ij} \oplus {}_{ij}R(i+j)_{ji} \oplus {}_{ji}R(i+j)_{ji}$$



$\cong$ 2×2 matrices over $\mathbb{Z}[x,y]$ with usual product except

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = xy \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = xy \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- $\nu = 2i + j$ $\qquad i — j$

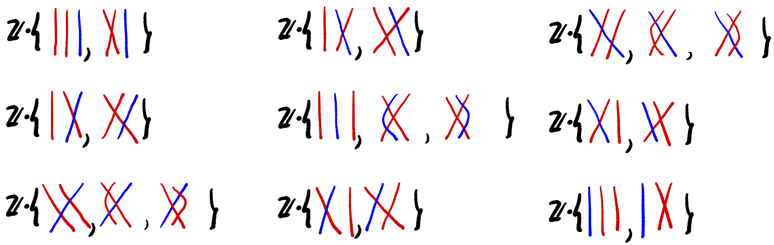
$\vec{i} = iij \qquad \vec{i} = iji \qquad \vec{i} = jii$

$_{\vec{j}}R(2i+j)_{\vec{i}} =$

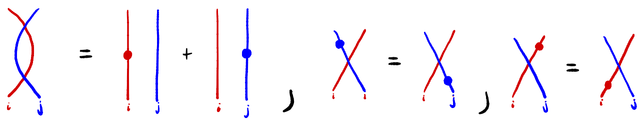
$\vec{j} = iij$

$\vec{j} = iji$

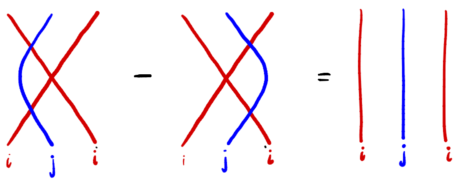
$\vec{j} = jii$



(+ dots)

- 

- NilHecke (for one color)

- 

(others : equal)

6. Crucial example: the NilHecke algebra

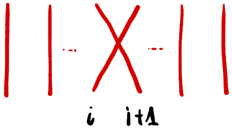
Consider the case $\mathfrak{g} = \mathfrak{sl}_2$, or $\Gamma = \bullet_i$

Then $\mathbf{f} = U_q^-(\mathfrak{sl}_2)$ should be categorified by $\bigoplus_{m \in \mathbb{Z}_{\geq 0}} R(mi)\text{-mod}$

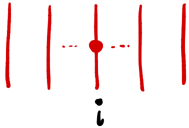
One feature was its canonical basis. In this case it is simply $\theta^{(m)}$, for $m \geq 0$.

$\rightsquigarrow$ We expect an indecomposable projective for each $m \geq 0$.

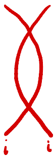
Denote $\color{red}{NH_m} = R(m\,i) = {}_{i\ldots i}R(m\,i)_{i\ldots i}$ = ring generated by

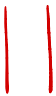
$u_i =$ 
deg -2

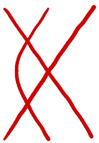
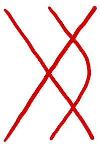
and

$x_i =$ 
deg 2

subject to isotopy and:

 $= 0$

 $-$  $=$  $=$  $-$ 

 $=$ 

$\longleftrightarrow$

x_i commute

$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

$u_i u_j = u_j u_i$ if $|i-j| > 2$

$x_i u_j = u_j x_i$ $|i-j| > 2$

$x_i u_{i+1} - u_{i+1} x_{i+1} = 1$

$x_{i+1} u_{i+1} - u_{i+1} x_i = 1$

The Nil Hecke algebra NH_m acts on $P_m = \mathbb{Z}[x_1, \dots, x_m]$ via

$$X_i \cdot f := x_i f$$

$$u_i \cdot f := \partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} \quad \text{where } s_i \text{ exchanges } x_i \leftrightarrow x_{i+1}$$

Demazure operators

For $w = s_{j_1} \cdots s_{j_\ell}$ a reduced expression, define $\partial_w = \partial_{s_{j_1}} \cdots \partial_{s_{j_\ell}}$

We will understand NH_m through its action on P_m.

It will be convenient to see P_m as a free $P_m^{S_m}$-module.

Fact: $\mathbb{Z}[x_1,\dots,x_m] \cong \mathcal{H}_m \otimes \underbrace{\mathbb{Z}[x_1,\dots,x_m]^{S_m}}_{Sym_m}$

where $\mathcal{H}_m = \mathbb{Z}\cdot\{x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdots x_m^{\alpha_m} : \alpha_i \leq m-i\}$. Note that $\mathrm{rk}\,\mathcal{H}_m = m!$

Example: $\mathbb{Z}[x_1,x_2,x_3] = \underbrace{\mathbb{Z}\cdot\{1,\ x_1,\ x_2,\ x_1x_2,\ x_2^2,\ x_1x_2^2\}}_{\mathcal{H}_3} \otimes \mathbb{Z}[x_1,x_2,x_3]^{S_3}$

The action gives a homomorphism $\varphi: NH_m \longrightarrow \mathrm{End}_{Sym_m}(P_m)$

Fact: $\mathrm{rk}_q(\mathcal{H}_m) = q^{\binom{m}{2}}[m]!$

Thus, $P_m \cong \bigoplus_{q^{\binom{m}{2}}[m]!} Sym_m$ and $\mathrm{End}_{Sym_m}(P_m) \cong M_{q^{\binom{m}{2}}[m]!}(Sym_m)$

as graded rings

Convenient basis: H_m has an integral basis given by the **Schubert polynomials:**

$$b_w := \partial_{w^{-1}w_0} x_1^{m-1} x_2^{m-2} \cdots x_{m-1}^{1}$$

Rmk: these represent unique homogeneous representatives for cohomology classes of Schubert varieties

Example: $m=3$, $w = s_1 s_2$

$$b_w = \partial_{s_2 s_1 s_1 s_2 s_1} x_1^2 x_2 = \partial_{s_1} x_1^2 x_2 = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = x_1 x_2$$

Remark: s_λ is a particular case.

Observe: $\partial_u b_w = \partial_u \partial_{w^{-1}w_0} x_1^{m-1} x_2^{m-2} \cdots x_{m-1}^{1} = \begin{cases} b_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise} \end{cases}$

We are ready to prove:

Proposition: $NH_m \longrightarrow M_{q^{\binom{m}{2}}[m]!}(Sym_m)$ is an isomorphism

$$\partial_u b_w = \begin{cases} b_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{otherwise} \end{cases}$$

Proof:

- Injectivity: assume some $\varphi\Big(\sum_{w\in S_m} \underbrace{f_w}_{\text{polynomials in } x_1,\dots,x_m} u_w\Big) = 0$. I.e., $\sum_{w\in S_m} f_w \partial_w = 0$

  In particular, if v_0 is of minimal length in the sum, $\sum_{w\in S_m} f_w \underbrace{\partial_w(b_{v_0})}_{b_{v_0 w^{-1}} \text{ if } \ell(v_0 w) = \ell(v_0) - \ell(w)} = 0$

  Minimality $\Rightarrow$ the contributions come from w with $\ell(w) = \ell(v_0)$ $\Rightarrow \ell(v_0 w) = 1 \Rightarrow w = v_0^{-1}$
  $\Rightarrow f_{v_0} = 0$ $\#$.

- Surjectivity: $\varphi(b_v \partial_{w_0}) = \begin{array}{c|c} & w_0 \\ \\ v & \end{array}\left(\begin{array}{cc} & 0 \\ 0 & \vdots \\ & 0 \\ & 1 \\ & 0 \\ & \vdots \\ & 0 \end{array}\right)$

  $\varphi(b_v \partial_{w_0 s_i}) = \begin{array}{c} \\ v \\ \\ s_i \\ \\ \end{array}\left(\begin{array}{ccc} & w & w_0 \\ & 0 & 0 \\ & 1 & \vdots \\ 0 & 0 & 0 \\ & \vdots & 1 \\ & & 0 \\ & 0 & 0 \end{array}\right) \longrightarrow$ subtract $b_{s_i} \partial_{w_0}$

  $\vdots$

$\square$

Corollary:

- $Z(NH_m) = Sym_m$

- $e = b_{w_0} \partial_{w_0} \mapsto \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ is a primitive idempotent

  $eNH_m \cong P_m$ (ungraded isom) is the unique projective module of NH_m $\leftrightarrow \theta_m$

  $NH_m \cong P_m^{\oplus \binom{m}{2}[m]!}$ as graded right NH_m-modules

- P_m / P_m^+ is the unique simple graded $k \otimes NH_m$-module for any field k.

7. A faithful representation of $R(\nu)$

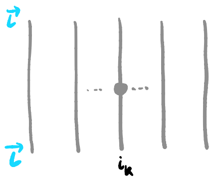
We can define an analog of the polynomial representation for every KLR algebra $R(\nu)$!

First fix an orientation on Γ. Define:

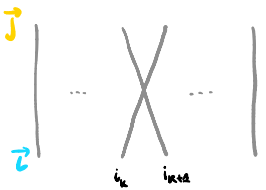
$$\mathrm{Pol}_\nu = \bigoplus_{\vec{i} \in \mathrm{Seq}(\nu)} \mathrm{Pol}_{\vec{i}} \qquad \mathrm{Pol}_{\vec{i}} = \mathbb{Z}[x_1(\vec{i}), \dots, x_m(\vec{i})] \qquad m = |\nu|$$

Let S_m act on $\mathrm{Pol}_{\vec{i}}$ by: $w \cdot (x_1(\vec{i})) = x_{w(1)}(w(\vec{i}))$

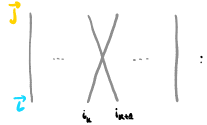
Define the action of the generators of ${}_{\vec{j}}R(\nu)_{\vec{i}}$ on $\mathrm{Pol}_{\vec{i}}$ (it acts as 0 on $\mathrm{Pol}_{\vec{k}}$ for $\vec{k} \neq \vec{i}$)



: $f \longmapsto x_k(\vec{i}) f \in \mathrm{Pol}_{\vec{i}}$

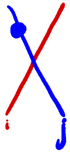
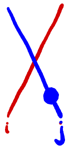


$$: f \longmapsto \begin{cases} \partial_{s_k} f & \text{if } i_k = i_{k+1} \\ s_k f & \text{if } i_k \not\!\!-\, i_{k+1} \text{ or } i_k \leftarrow i_{k+1} \\ (x_k(s_k\vec{i}) + x_{k+1}(s_k\vec{i}))(s_k f) & \text{if } i_k \rightarrow i_{k+1} \end{cases} \in \mathrm{Pol}_{\vec{j}}$$

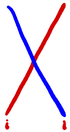
 $: f \longmapsto$

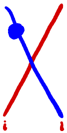
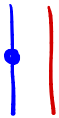
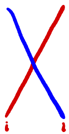
$(x_k(s_k \vec{i}) + x_{k+1}(s_k \vec{i}))(s_k f)$ if $i_k \to i_{k+1}$

Example: $\nu = i + j$ $\quad$ $i \to j$

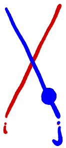
 $=$ 

let $f \in Pol_{ij} = \mathbb{Z}[x_1(ij), x_2(ij)]$

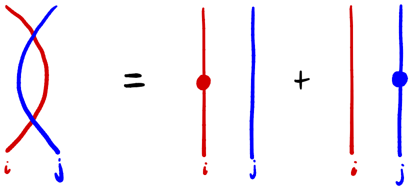
 $f = (x_1(s \cdot ij) + x_2(s \cdot ij))(sf) = (x_1(ji) + x_2(ji))(sf)$

 $f =$  $\bullet$  $f = x_1(ji) \cdot (x_1(ji) + x_2(ji))(sf)$

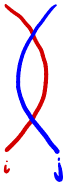
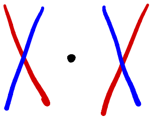
On the other hand,

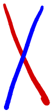
 $f = (x_1(s \cdot ij) + x_2(s \cdot ij)) \cdot (s(x_2(ij)f)) = (x_1(ji) + x_2(ji))\, x_1(ji) \cdot sf$

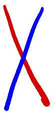
$\nu = i + j \qquad i \to j$



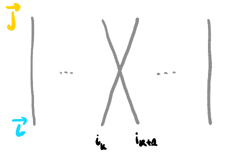
Notation: we may drop $\vec{\imath}$ from $x_k(\vec{\imath})$

 f =  f

$=$  $(x_1 + x_2)(sf)$

$=$  $s(x_1 + x_2)(s^2 f)$

$= (x_2 + x_1)\, f$

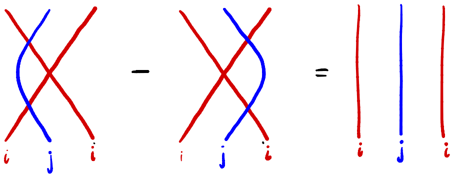


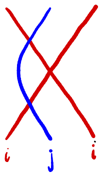
$: f \longmapsto$

$$\begin{cases} s_k f & \text{if } i_k \not\!\!-\, i_{k+1} \text{ or } i_k \leftarrow i_{k+1} \\ \left(x_k(s_k\vec{\imath}) + x_{k+1}(s_k\vec{\imath})\right)(s_k f) & \text{if } i_k \to i_{k+1} \end{cases}$$

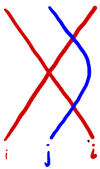
$$\begin{cases} \partial_k f & \text{if} \quad i_k = i_{k+1} \\ s_k f & \text{if} \quad i_k \not\!\!-\!\!-\, i_{k+1} \quad \text{or} \quad i_k \leftarrow i_{k+1} \\ \left(x_k(s_k \bar{i}) + x_{k+1}(s_k \bar{i})\right)(s_k f) & \text{if} \quad i_k \rightarrow i_{k+1} \end{cases}$$

Example: $\nu = 2i + j \qquad i \rightarrow j$

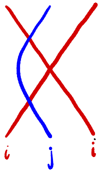




$$f = s_1 \partial_{s_2}\left((x_1 + x_2)(s_1 f)\right)$$



$$f = (x_1 + x_2)\, s_2\left(\partial_{s_1}(s_2 f)\right)$$

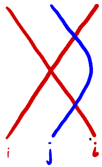


$$x_1^a x_2^b x_3^c = s_1 \partial_{s_2}\Big((x_1+x_2)\,(s_1 x_1^a x_2^b x_3^c)\Big)$$

$$= s_1 \partial_{s_2}\Big((x_1+x_2)\,(x_1^b x_2^a x_3^c)\Big)$$

$$= s_1 \, \frac{x_1^{b+1} x_2^a x_3^c - x_1^{b+1} x_2^c x_3^a + x_1^b x_2^{a+1} x_3^c - x_1^b x_2^c x_3^{a+1}}{x_2 - x_3}$$

$$= \frac{x_1^a x_2^{b+1} x_3^c - x_1^c x_2^{b+1} x_3^a + x_1^{a+1} x_2^b x_3^c - x_1^c x_2^b x_3^{a+1}}{x_1 - x_3}$$

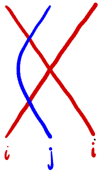


$$x_1^a x_2^b x_3^c = (x_2+x_3)\, s_2\Big(\partial_{s_1}(s_2 x_1^a x_2^b x_3^c)\Big)$$

$$= (x_2+x_3)\, s_2 \, \frac{x_1^a x_2^c x_3^b - x_1^c x_2^a x_3^b}{x_1 - x_2}$$

$$= (x_2+x_3) \, \frac{x_1^a x_2^b x_3^c - x_1^c x_2^b x_3^a}{x_1 - x_3}$$

$$= \frac{x_1^a x_2^{b+1} x_3^c - x_1^c x_2^{b+1} x_3^a + x_1^a x_2^b x_3^{c+1} - x_1^c x_2^b x_3^{a+1}}{x_1 - x_3}$$

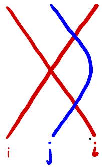


$$x_1^a x_2^b x_3^c = s_1 \partial_{s_2}\left(\left(x_1 + x_2\right)\left(s_1 x_1^a x_2^b x_3^c\right)\right)$$

$$= s_1 \partial_{s_2}\left(\left(x_1 + x_2\right)\left(x_1^b x_2^a x_3^c\right)\right)$$

$$= s_1 \frac{x_1^{b+1} x_2^a x_3^c - x_1^{b+1} x_2^c x_3^a + x_1^b x_2^{a+1} x_3^c - x_1^b x_2^c x_3^{a+1}}{x_2 - x_3}$$

$$= \frac{x_1^a x_2^{b+1} x_3^c - x_1^c x_2^{b+1} x_3^a + x_1^{a+1} x_2^b x_3^c - x_1^c x_2^b x_3^{a+1}}{x_1 - x_3}$$



$$x_1^a x_2^b x_3^c = \left(x_2 + x_3\right) s_2\left(\partial_{s_1}\left(s_2 x_1^a x_2^b x_3^c\right)\right)$$

$$= \left(x_2 + x_3\right) s_2 \frac{x_1^a x_2^c x_3^b - x_1^c x_2^a x_3^b}{x_1 - x_2}$$

$$= \left(x_2 + x_3\right) \frac{x_1^a x_2^b x_3^c - x_1^c x_2^b x_3^a}{x_1 - x_3}$$

$$= \frac{x_1^a x_2^{b+1} x_3^c - x_1^c x_2^{b+1} x_3^a + x_1^a x_2^b x_3^{c+1} - x_1^c x_2^b x_3^{a+1}}{x_1 - x_3}$$

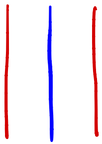
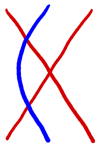
Difference: $x_1^a x_2^b x_3^c$ as desired

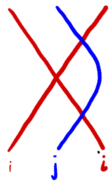
8. A basis for ${}_{\vec{j}}R(\nu)_{\vec{i}}$

- It's clear that we can slide dots to the bottom
- Whenever two strands intersect twice, we can simplify the diagram.

Idea: choose a reduced expression for each $w \in S_n$ taking $\vec{i} \mapsto \vec{i}$, $\vec{j} \mapsto \vec{j}$ and form the corresponding diagrams ${}_{\vec{j}}\hat{S}_{\vec{i}}$

${}_{iji}R(2i+j)_{iji}$ $\qquad \underline{1}$, $\underline{(13)} = s_1 s_2 s_1$

$${}_{iji}\hat{S}_{iji} = \Big\{ \hat{1} =$$  $$, \quad \widehat{(13)} =$$  $$\Big\}$$

Remark: Had we chosen $\underline{(13)} = s_2 s_1 s_2$, $\widehat{(13)}$ would have been 

Denote ${}_{\mathbf{j}}B_{\mathbf{i}} = \{$ Diagrams in ${}_{\mathbf{j}}\hat{S}_{\mathbf{i}}$ with dots on the bottom $\}$

Theorem: ${}_{\mathbf{j}}R(\nu)_{\mathbf{i}}$ is a free graded abelian group with basis ${}_{\mathbf{j}}B_{\mathbf{i}}$

Idea:
- ${}_{\mathbf{j}}B_{\mathbf{i}}$ is homogeneous and it spans ${}_{\mathbf{j}}R(\nu)_{\mathbf{i}}$
- Choose orientations on Γ and order I so that
- Let ${}_{\mathbf{j}}R(\nu)_{\mathbf{i}}$ act : $Pol_{\mathbf{i}} \to Pol_{\mathbf{j}}$ so that $i \to j$ implies $i < j$.
- The order gives a lexicographic order on $Seq(\nu)$
- Induction on this order to show the operators $Pol_{\mathbf{i}} \to Pol_{\mathbf{j}}$ are linearly independent.

  (Base case follows from the analogous result for Demazure operators)

Corollary: Pol_ν is a faithful representation of $R(\nu)$.

Thank you!

Questions?