Induction and Restriction

Sunday, January 30, 2022 11:18 AM

- Given $i: B \hookrightarrow A$, $i(1) \neq 1$, $i(1)=e$ only idempotent

<u>Exer 1</u>: (a) $\mathrm{Ind}: B\text{-mod} \to A\text{-mod}$ (all left modules)

$$\mathrm{Ind}(M) \cong Ae \otimes_B M$$

(b) $\mathrm{Res}: A\text{-mod} \to B\text{-mod}$

$$\mathrm{Res}(N) \cong eN$$

For KLR algebras, $\otimes = \boxtimes := \otimes_{\mathbb{Z}}$

$$i_{v,w}: R(\alpha) \otimes R(\beta) \hookrightarrow R(\alpha+\beta)$$

$$\boxed{A} \otimes \boxed{B} \longmapsto \boxed{A \,|\, B}$$

is not necc unital, as

$$e_{v,w} := 1_{R(\alpha)\otimes R(\beta)} = \sum_{\substack{\vec{i} \in Seq(\alpha) \\ \vec{j} \in Seq(\beta)}} 1_{\vec{i}} \otimes 1_{\vec{j}} \neq 1_{R(\alpha+\beta)}$$

$1_{\vec{i}} \otimes 1_{\vec{j}} = |_{\vec{i}}\,|_{\vec{j}}$, $1_{\vec{i},\vec{j}}$

if $\exists \vec{u} \in Seq(\alpha+\beta)$ not in form $Seq(\alpha)\otimes Seq(\beta)$

<u>Ex:</u> $\alpha = i$, $\beta = j$, $\vec{u} = (j, i)$

<u>Prop 2:</u> $e_{\alpha,\beta} R(\alpha+\beta)$ is a free graded left $R(\alpha)\otimes R(\beta)$-mod.

<u>Pf:</u> $e_{\alpha,\beta} R(\alpha+\beta) = \left\{ \begin{array}{l} \text{diagrams ending in } (\vec{i},\vec{j}) \\ \vec{i} \in Seq(\alpha),\ \vec{j} \in Seq(\beta) \end{array} \right\}$

$R(\alpha)\otimes R(\beta) = \left\{ \begin{array}{l} \text{diagrams starting at } (\vec{\ell},\vec{k}), \text{ end at } (\vec{i},\vec{j}), \\ \vec{\ell}, \vec{i} \in Seq(\alpha),\ \vec{k}, \vec{j} \in Seq(\beta) \end{array} \right\}$

w/ action given by

$\boxed{\quad\ \ |\quad\ \ }$ $\in R(\alpha)\otimes R(\beta)$

$i_1\ i_2\ \ j_1\ j_2$

$\tau_{(23)} = |\ \times\ | \in e_{\alpha,\beta} R(\alpha+\beta)$

Fix $(\vec{i},\vec{j}) \in Seq(\alpha)\otimes Seq(\beta)$. Recall ${}_{(\vec{i},\vec{j})}R(\alpha+\beta)_{\vec{m}}$ has a $\mathbb{Z}$-basis $\left\{ 1_{(\vec{i},\vec{j})} x_1^{n_1} \cdots x_{|v|+|w|}^{n_{|v|+|w|}} \tau_w 1_{\vec{m}} \right\}_{n_i \in \mathbb{Z}^+}^{w \in S_n}$

$\tau_w =$ Choose some reduced exp $\underline{w}$ for $w \in S_n$ represent it pictorially, then add labels $(\vec{i},\vec{j})$ det by $(\vec{i},\vec{j})$ and w

Induction and Restriction 2

Sunday, January 30, 2022 11:18 AM

Note • $1_{(\vec{i},\vec{j})} x_1^{n_1} \cdots x_{|\alpha|+|\beta|}^{n_{|\alpha|+|\beta|}} \in R(\alpha) \otimes R(\beta)$

• $\tau_w \in R(\alpha) \otimes R(\beta) \iff w \in S_{|\alpha|} \times S_{|\beta|} \subseteq S_{|\alpha|+|\beta|}$

$\Rightarrow \{1_{(\vec{i},\vec{j})} \tau_{\bar{w}}\}_{\bar{w} \in S_{|\alpha|} \times S_{|\beta|} \backslash S_{|\alpha|+|\beta|}}$ some rep

spans as $R(\alpha) \otimes R(\beta)$-mod

Q: Is it L.I. over $R(\alpha) \otimes R(\beta)$?

A: No, $1_{(\vec{i}',\vec{j}')} (1_{(\vec{i},\vec{j})} \tau_{\bar{w}}) = 0$ for $\vec{i}' \neq \vec{i}$

$\rightsquigarrow$ Consider

$$\hat{\tau}_{\bar{w}} = \sum_{\substack{\vec{\ell} \in Seq(\alpha) \\ \vec{k} \in Seq(\beta)}} 1_{(\vec{\ell},\vec{k})} \tau_{\bar{w}}$$

- Note $1_{(\vec{i},\vec{j})} \hat{\tau}_{\bar{w}} = 1_{(\vec{i},\vec{j})} \tau_{\bar{w}}$

$\Rightarrow \{\hat{\tau}_{\bar{w}}\}_{\bar{w} \in S_{|\alpha|} \times S_{|\beta|} \backslash S_{|\alpha|+|\beta|}}$ some rep spans

Q: Is it L.I. over $R(\alpha) \otimes R(\beta)$?

A: No, can't use random rep

Exer 3 (a) For each right coset $S_{|\alpha|} \times S_{|\beta|} \backslash S_{|\alpha|+|\beta|}$ $\exists$ rep of minimal length. Let MCR = set of all min length coset rep. Show $\forall w \in S_{|\alpha|+|\beta|}$ $\exists!$

$$w = uv, \quad u \in S_{|\alpha|} \times S_{|\beta|}, v \in MCR$$

(b) Show $\{\hat{\tau}_{\bar{w}}\}_{\bar{w} \in MCR}$ is L.I over $R(\alpha) \otimes R(\beta)$

Let $Ind_{\alpha,\beta}^{\alpha+\beta}$, $Res_{\alpha,\beta}^{\alpha+\beta}$ be Ind, Res corr to

$$i_{\alpha,\beta} : R(\alpha) \otimes R(\beta) \longrightarrow R(\alpha+\beta)$$

Exer 1 + Prop 2 $\Rightarrow$

Cor 4: $Ind_{\alpha,\beta}^{\alpha+\beta}$, $Res_{\alpha,\beta}^{\alpha+\beta}$ sends proj to proj

Cor 5: $Ind_{\alpha,\beta}^{\alpha+\beta}$, $Res_{\alpha,\beta}^{\alpha+\beta}$ descend to

$$[Ind_{\alpha,\beta}^{\alpha+\beta}] : K_{\oplus}(R(\alpha)) \otimes K_{\oplus}(R(\beta)) \longrightarrow K_{\oplus}(R(\alpha+\beta))$$

$$[Res_{\alpha,\beta}^{\alpha+\beta}] : K_{\oplus}(R(\alpha+\beta)) \longrightarrow K_{\oplus}(R(\alpha)) \otimes K_{\oplus}(R(\beta))$$

Bialgebra Structure

Sunday, January 30, 2022 11:18 AM

and similarly with $K_0(R(\alpha))$

$K_\oplus(R(\alpha)) := K_\oplus(R(\alpha)\text{-mod})$

$K_0(R(\alpha)) := K_0(R(\alpha)\text{-mod})$

Pf: Exer1 + Prop 1 $\Rightarrow \mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}, \mathrm{Res}_{\alpha,\beta}^{\alpha+\beta}$ exact

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Given simply laced Γ, let

$$R_\Gamma = \bigoplus_{\alpha\in Q_\Gamma^+} R(\alpha)$$

$\rightsquigarrow K_\oplus(R_\Gamma) = \bigoplus_{\alpha\in Q_\Gamma^+} K_\oplus(R(\alpha)), \; K_0(R_\Gamma) = \bigoplus_{\alpha\in Q_\Gamma} K_0(R(\alpha))$

Def Ind: $R_\Gamma\otimes R_\Gamma\text{-mod} \to R_\Gamma\text{-mod}$, $V\in R_\alpha\text{-mod}$, $W\in R_\beta\text{-mod}$

$$V\circ W = \mathrm{Ind}(V\boxtimes W) = \mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}(V\boxtimes W)$$

Res: $R_\Gamma\text{-mod} \to R_\Gamma\otimes R_\Gamma\text{-mod}$, $U\in R_\gamma\text{-mod}$

$$\mathrm{Res}(U) = \bigoplus_{\alpha+\beta=\gamma} \mathrm{Res}_{\alpha,\beta}^{\gamma}(U)$$

- By Cor 5, [Ind], [Res] descend to $K_\oplus, K_0$

Prop 6: [Res], [Ind] turns $K_\oplus(R_\Gamma)$ into (co)associative, (co)unital (co)algebras, respectively.

Similarly with $K_0(R_\Gamma)$.

Pf: unit $= (R(0)=\mathbb{Z})\text{-mod } \mathbb{Z}$. counit$(V) = \mathrm{grd}_{\mathbb{Z}} V$

Prop 7: $[\mathrm{Res}]: K_0(R_\Gamma) \to K_0(R_\Gamma)\otimes K_0(R_\Gamma)$ is an algebra homomorphism w/ twisted alg structure on $K_0(R_\Gamma)\otimes K_0(R_\Gamma)$ given by

$$([a]\otimes[b]) \star ([c]\otimes[d]) = q^{-\beta\cdot\gamma}([a\circ c]\otimes[b\circ d])$$

$b\in R_\beta\text{-mod}$, $c\in R_\gamma\text{-mod}$ $\Rightarrow K_0(R_\Gamma)$ is a bialg

Pf: Will just show for $\Gamma = \cdot \Rightarrow Q_\Gamma = \mathbb{Z}^{\geq 0}$, $R_n = NH_n$. WTS

$$\mathrm{Res}([V]\circ[W]) = \mathrm{Res}[V] \star \mathrm{Res}[W]$$

$$\parallel$$

$$\mathrm{Res}(\mathrm{Ind}_{NH_n\times NH_m}^{NH_{n+m}} [V]\otimes[W])$$

Bialgebra Structure 2

Sunday, January 30, 2022 11:18 AM

Lemma 8: Given a representation L of $S_n \times S_m$

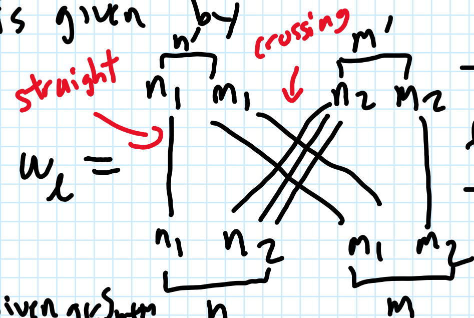
$$\mathrm{Res}^{n+m}_{n',m'}\mathrm{Ind}^{n+m}_{n,m}(L) \simeq \bigoplus \mathrm{Ind}^{n',m'}_{n_1,m_1,n_2,m_2}\mathrm{Res}^{n,m}_{n_1,n_2,m_1,m_2}(L)$$

$n_1+n_2=n,\ m_1+m_2=m$ switched!

- $n'+m'=n+m$ $\quad n_1+m_1=n',\ n_2+m_2=m'$
- n_1,m_1,n_2,m_2 means $S_{n_1}\times S_{m_1}\times S_{n_2}\times S_{m_2}$

Pf: Claim that a set of coset rep of $S_{n'}\times S_{m'} \backslash S_{n+m} / S_n\times S_m$ is given by

$w_\ell =$



- ℓ = # of crossing lines from $\min(n,m,n',m')$ region
- Here • $n=4, m=3$ • $n_1=1, n_2=3$ • $n'=3, m'=4$ • $m_1=2, m_2=1$ • $\ell=2$

Given $g\in S_{n+m}$

Algor: (1) Use $S_{n'}$-action to bring all lines from n'-region to m region of g into the (rightmost region of n') $=: m_1$

(2) Use S_m-action to bring all m_1 crossing lines to m into leftmost region of m. ⇝ creates region m_2 of m

(3) Use $S_{m'}$-action to bring all lines from m (m_2) into m' to rightmost of m' ⇝ creates region n_2

(4) Use S_n-action to bring all lines from n' to rightmost of n ⇝ creates region n_1

(5) Use $S_{n_1}\times S_{n_2}\times S_{m_1}\times S_{m_2}$-action to straighten everything out.

Plug this into Mackey iso ∎

Rem: If L is a rep of NH_{n+m}, L8 not true as NH_{n+m} is not s.s. However true after passing to K_0! Sketch: Interpret $w_\ell \in NH_{n+m}$ ⇝ construct a filtration on

$$\mathrm{Res}^{NH_{n+m}}_{NH_{n'}\times NH_{m'}}\mathrm{Ind}^{NH_{n+m}}_{NH_n\times NH_m}(L)$$

w/ associated graded, analogue of RHS of L8 + grading shifts

- Finally let $L = V\boxtimes W$

$$\mathrm{Res}^{n,m}_{n_1,n_2,m_1,m_2}(V\boxtimes W) \simeq \mathrm{Res}^{n}_{n_1,n_2}(V)\boxtimes \mathrm{Res}^{m}_{m_1,m_2}(W)$$

- Switching in L8 corresponds to twisted alg structure on $K_0(\mathcal{R}_p)$

Hopf Pairing

Sunday, January 30, 2022 11:18 AM

Rem: If L is proj, then L8 holds b/c Ind, Res send proj to proj so the filtration consists of proj mod $\Rightarrow$ splits. Thus Prop 7 is also true for $K_{\oplus}(R_P)$

Lemma 9: (a) $\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}(P_{\vec{i}} \boxtimes P_{\vec{j}}) \cong P_{\vec{i}\vec{j}}$ as $1_{\vec{i}} \boxtimes 1_{\vec{j}} \longmapsto 1_{(\vec{i},\vec{j})}$. Also true for $(\vec{i}) \in \mathrm{Seq}(\alpha)$

(b) Let $\vec{i} = i$. Then

$$\mathrm{Res}\, P_i = \mathrm{Res}_{1,0}^{1} P_i \oplus \mathrm{Res}_{0,1}^{1} P_i \cong P_i \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes P_i$$

Recall Cartan pairing: $K_{\oplus}(R(\alpha)) \times K_0(R(\alpha)) \to \mathbb{Z}[q^{\pm 1}]$

$$([P],[M])_c := \mathrm{gdim}_{\Bbbk}(P^{\psi} \otimes_{R(\alpha)} M)$$

Extend to $K_{\oplus}(R_P) \times K_0(R_P) \to \mathbb{Z}[q^{\pm 1}]$ by

$$([a],[b])_c = 0 \quad \text{if } a \in R(\alpha)\text{-pmod}, \beta \in R(\beta)\text{-mod}$$
$$\alpha \neq \beta$$

- Recall $([P_b],[S_a])_c = \delta_{ab} \Rightarrow (\,,\,)_c$ is non-deg

$\Rightarrow K_0(R_P) \cong K_{\oplus}(R_P)^*$

Prop 10: Restrict $(\,,\,)_c$ on RHS to $K_{\oplus}(R_P)$

(1) $([R(0)],[R(0)])_c = 1$

(2) $([P_i],[P_j])_c = \dfrac{\delta_{ij}}{(1-q^2)}$ $\forall i,j \in I_P$

(3) $([M],[N_1]\circ[N_2])_c = ([\mathrm{Res}](m), [M] \otimes [N_2])_c$

(4) $(\,,\,)_c$ is symmetric

Pf: (1), (4) easy

(2) $\mathrm{gdim}_{\Bbbk}({}_iP \otimes_{R(i)} P_i) = \mathrm{gdim}_{\Bbbk}({}_iR(i)_i)$

$= \mathrm{gdim}_{\Bbbk}(\Bbbk[x]) = \dfrac{1}{1-q^2}$

(3) $M \in R(\alpha+\beta)$-pmod, $N_1 \in R(\alpha)$-mod, $N_2 \in R(\beta)$-mod

$([M],[N_1]\circ[N_2]) = ([M], [\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} N_1 \boxtimes N_2])$

$= \mathrm{gdim}_{\Bbbk}(M^{\psi} \otimes_{\alpha+\beta} R_{\alpha+\beta} e_{\alpha,\beta} \otimes_{\alpha,\beta} N_1 \boxtimes N_2)$

$= \mathrm{gdim}_{\Bbbk}(M^{\psi} e_{\alpha,\beta} \otimes_{\alpha,\beta} N_1 \boxtimes N_2)$, $\psi(e_{\alpha,\beta}) = e_{\alpha,\beta}$

$= \mathrm{gdim}_{\Bbbk}(\mathrm{Res}\, M)^{\psi} \otimes_{\alpha,\beta} N_1 \boxtimes N_2)$

$$= \operatorname{gdim}_{\Bbbk}\left(M^{\tau} e_{\alpha,\beta} \otimes_{\alpha,\beta} N_1 \boxtimes N_2\right)$$

$$= \operatorname{gdim}_{\Bbbk}\left((e_{\alpha,\beta} M)^{\tau} \otimes_{\alpha,\beta} N_1 \boxtimes N_2\right), \quad \tau(e_{\alpha,\beta}) = e_{\alpha,\beta}, \ \tau \text{ anti-auto}$$

$$= \left(\left[\operatorname{Res}^{\alpha+\beta}_{\alpha,\beta} M\right], [N_1] \boxtimes [N_2]\right)$$

Recall. $f^{\Gamma} = \Bbbk\langle \theta_i \rangle_{i \in I_\Gamma}$ / quantum Serre relations,

- $f^{\Gamma}_{\mathbb{Z}[q^{\pm 1}]} = \mathbb{Z}[q^{\pm 1}]$-subalg gen by $\theta_i^{(n)} = \frac{\theta_i^n}{[n]!}$

Thm 11 We have an isomorphism of twisted bialgebras

$$\gamma: f^{\Gamma}_{\mathbb{Z}[q^{\pm 1}]} \xrightarrow{\sim} K_{\oplus}(R_\Gamma)$$

where $\gamma(\theta_{i_1}^{(a_1)} \cdots \theta_{i_k}^{(a_k)}) = [P_{\vec{i}}]$, $\vec{i} = (i_1^{(a_1)}, \ldots, i_k^{(a_k)})$. Moreover,

(a) $(x, y)_{\text{Drinfeld}} = (\gamma(x), \gamma(y))_C$

(b) $\gamma(\bar{x}) = \overline{\gamma(x)} =: \mathrm{HOM}_{R_\Gamma}(\gamma(x), R_\Gamma)^{\psi}$

Pf: γ is a homomorphism / Recall $[P_{ij}] = [P_{ji}]$ $i \cdot j = 0$

$[P_{iji}] = [P_{i^{(2)}j}] + [P_{ji^{(2)}}]$ $i \cdot j = -1$

$\Rightarrow \gamma$ is an alg homomorphism

In $f^{\Gamma}_{\mathbb{Z}[q^{\pm 1}]}$, $\Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$

- the "K_i" part is accounted by the twisted alg structure on $f^{\Gamma} \otimes f^{\Gamma}$ should be K_i??

- matches w/ corr eq for $[\mathrm{Res}](P_i)$
- both Δ and $[\mathrm{Res}]$ are alg homo, agree on gen $\Rightarrow \gamma$ is a coalg homomorph

1-1 Lemma 12 (Drinfeld): $\exists!$ pairing $(,)_D$ on $f^{\Gamma}_{\mathbb{Z}[q^{\pm 1}]}$ characterized by (1)–(4) in Prop 10

Moreover $(,)_D$ is non-deg

$\Rightarrow$ (a) $(x,y)_D = (\gamma(x), \gamma(y))_C$ as γ homomorph

$\Rightarrow \gamma$ is injective by non-deg of $(,)_D$

(b) Both $\bar{\ }$ in f and $K_{\oplus}(R_\Gamma)$ are q-antilinear alg automorph fixing gen $\bar{\theta_i} = \theta_i$, $\bar{P_i} = P_i$

$\Rightarrow \gamma(\bar{x}) = \overline{\gamma(x)}$

Categorification: Surjectivity

Sunday, January 30, 2022 11:18 AM

Onto! **Def** $\mathrm{Ch}: R_p\text{-mod} \longrightarrow \mathbb{Z}[q,q^{-1}]\langle I_p \rangle$

$$\mathrm{Ch}(M) = \sum_{\alpha \in Q_p^+} \sum_{\vec{i} \in \mathrm{Seq}(\alpha)} \mathrm{gdim}_{\Bbbk}(1_{\vec{i}} M)\, \vec{i}$$

Prop 13 (KL): Ch is injective

Pf Sketch: Show $\{\mathrm{Ch}(L_i)\}^{L_i \text{ simple}}$ is L.I. $\blacksquare$

Using $(\,,)_D$ nondeg, we have that

$$f^{p*}_{\mathbb{Z}[q^{\pm 1}]} = \{ y \in f^p \mid (x,y)_D \in \mathbb{Z}[q^{\pm 1}] \;\forall x \in f^p_{\mathbb{Z}[q^{\pm 1}]} \}$$

-Now dualize γ wrt $(\,,)_D$

$$\gamma^*: K_0(R_p) \longrightarrow f^{p*}_{\mathbb{Z}[q^{\pm 1}]}$$

aka $(\theta, \gamma^*([M]))_D = (\gamma(\theta), [M])_C$

Def $\chi: f^{p*}_{\mathbb{Z}[q^{\pm 1}]} \longrightarrow \mathbb{Z}[q,q^{-1}]\langle I_p \rangle$

$$\chi(y) = \sum_{\alpha \in Q_p^+} \sum_{\vec{i} \in \mathrm{Seq}(\alpha)} (\theta_{\vec{i}}, y)_D\, \vec{i}$$

Claim: The following diagram commutes

$$\begin{tikzcd}
K_0(R_p) \arrow[rr, "\gamma^*"] \arrow[dr, "\mathrm{Ch}"'] & & f^{p*}_{\mathbb{Z}[q^{\pm 1}]} \arrow[dl, "\chi"] \\
& \mathbb{Z}[q,q^{-1}]\langle I_p \rangle &
\end{tikzcd}$$

Pf: $(\theta_{\vec{i}}, \gamma^*[M])_D = (\gamma(\theta_{\vec{i}}), [M])_C$

$= (P_{\vec{i}}, [M])_C = \mathrm{gdim}_{\Bbbk}(\vec{i}\,P \otimes_{R_p} [M])$

$= \mathrm{gdim}_{\Bbbk}(1_{\vec{i}} M)$ $\blacksquare$

-Ch inj $\Rightarrow \gamma^*$ inj $\Rightarrow \gamma$ is surj

$\Rightarrow \gamma, \gamma^*$ are isomorphisms

Generalization to non simply laced

Sunday, January 30, 2022 11:18 AM

Let $C=(C_{ij})_{i,j\in I}$ be a Cartan matrix. Fix

$$Q_{ij}(u,v)=\sum_{k,m} Q_{ij}^{(k,m)}u^k v^m \in \mathbb{C}[u,v]$$

$\forall i,j$ s.t. $\quad |u|=d_i, |v|=d_j$

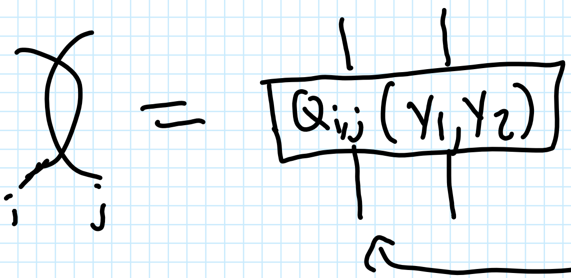
(1) $Q_{ii}=0$

(2) $Q_{ij}(u,v)$ is homogeneous (of deg $-d_iC_{ij}$)

(3) $Q_{ij}(u,v)=Q_{ji}(v,u)$

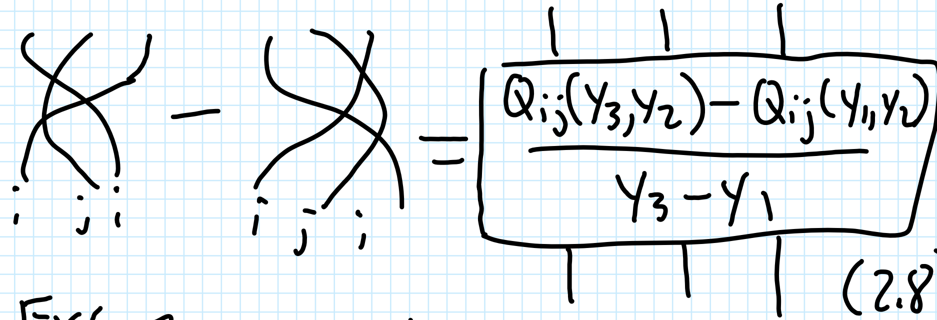
Then for $v\in\mathbb{N}[I]$, define

$R^C(v)=R(v)$ with modified local relations



(2.3)

$y_1^a=$ a dots on



(2.8)

Ex: $\Gamma=\underset{1}{\bullet}\rightarrow\underset{2}{\bullet}\leftarrow\underset{3}{\bullet}$

$Q_{12}=u-v,\ Q_{23}=v-u,\ Q_{13}=1$

$\rightsquigarrow$ recover signed version of R_Γ

Rem: $Q_{ij}(u,v)=\begin{cases}u+v & i\cdot j=-1\\ 1 & i\cdot j=0\\ 0 & i=j\end{cases}$ recovers R_Γ

Ex: $\Gamma=\underset{1}{\bullet}\Rrightarrow\underset{2}{\bullet}$, $d_1=1$, $d_2=2$, $\begin{pmatrix}2&-2\\-1&2\end{pmatrix}$

$Q_{12}=u^2+v$, $Q_{21}=v^2+u$ $\quad |u|=1$, $-d_1C_{12}=2$