1. Lusztig's quantum $\mathfrak{sl}_2$

Goal Categorify entire $U_q\mathfrak{sl}_2 =: \mathbb{U}$

Recall $\mathbb{U}$ is the $\mathbb{Q}(q)$-algebra with 1 generated by E, F, K, K^{-1}
subject to the relations • $KK^{-1} = 1 = K^{-1}K$

• $KE = q^2 EK$ • $KF = q^{-2} FK$ • $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

Beilinson-Lusztig-Macpherson

add orthogonal idempotents replacing $\mathbb{U}_0$ for projections $V \to V(n)$
to produce

$$\dot{\mathbb{U}} = \mathrm{span}_{\mathbb{Q}(q)}\{E^a F^b 1_n : a, b \geq 0, n \in \mathbb{Z}\}$$

relations become • $1_n 1_m = \delta_{n,m} 1_n$

• $E 1_n = 1_{n+2} E$ • $F 1_n = 1_{n-2} F$ • $EF1_n - FE1_n = [n] 1_n$

$[n] = q^{n-1} + \ldots + q^{1-n}$

"no more K"
$K 1_n = q^n 1_n$

Rem integral dot form spanned by divided powers, s.t. ∞ rels one for each power

Rem CF '94 conj'd $\dot{\mathbb{U}}_{\mathbb{Z}}$ could be categorified using Lusztig's

$$\dot{\mathbb{B}} = \{ E^{(a)}F^{(b)}1_n : b-a \geq n, a,b \geq 0 \} \cup \{ F^{(b)}E^{(a)} : a,b \geq 0, b-a \leq n \}$$

since str consts in $\mathbb{N}[q^{\pm}]$

in a pre-add cat hom sets are ab gps & composition is bilin.

Fact Algs w systs of idems = (small) pre-add cats
(or idem'd rings)

cf add cats where biprods are finite... $\sum 1_n \nexists$

and pre-add cats are K_0's (2-cats)

Therefore if our goal is a higher structure with Grothendieck ring given by $\dot{\mathbb{U}}$ we can expect it's a 2-category $\mathcal{U}$

$$\text{(small) pre-add 2-cats} \longleftrightarrow \text{idempot add monoidal cats}$$

$$\Big\downarrow K_0$$

$$\text{(small) pre-add cats} \longleftrightarrow \text{idempotented rings}$$

$\dot{\mathcal{U}}$ as pre-add cat:

objs

$n \in \mathbb{Z}$

mors

ab gp

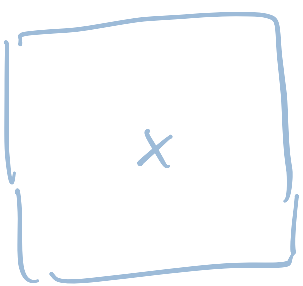
$\mathrm{Hom}(n,m) = 1_m \dot{\mathcal{U}} 1_n$

- $n=m \Rightarrow$ just id mor 1_n
- comp: $1_{m'} \dot{\mathcal{U}} 1_m \otimes 1_{n'} \dot{\mathcal{U}} 1_n \longrightarrow \delta_{n',m} 1_{m'} \dot{\mathcal{U}} 1_n$
  $=$ mult

2. Graphical Calculus for 2-categories

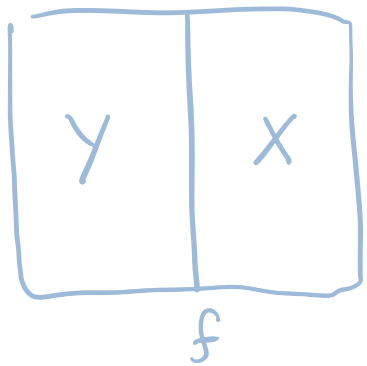
reading dirs

objs



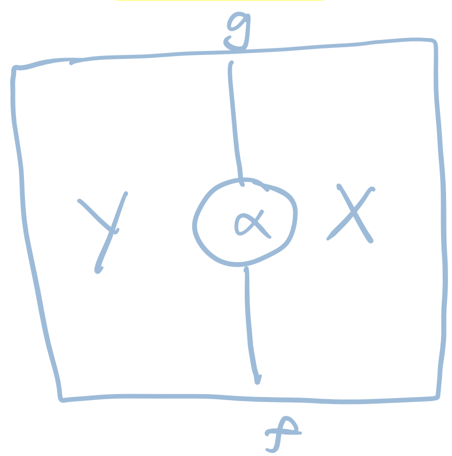
regions in the plane

mors



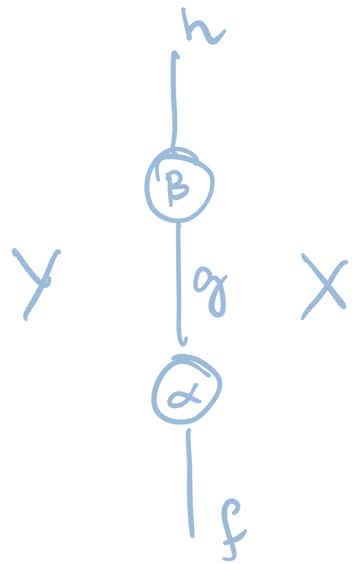
lines separating regions

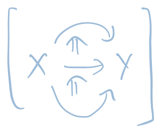
2-mors



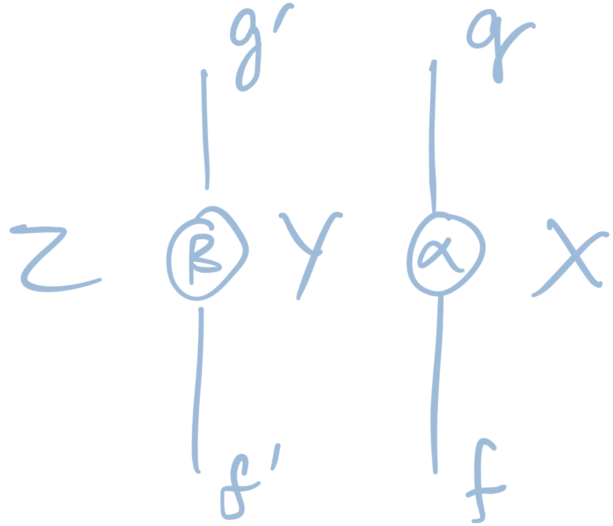
points separating lines

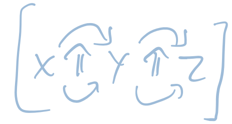
vertical comp of 2-mor



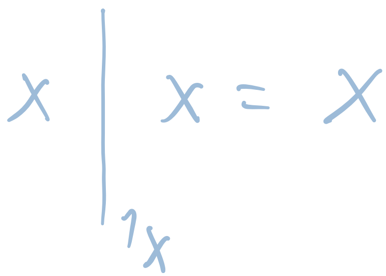


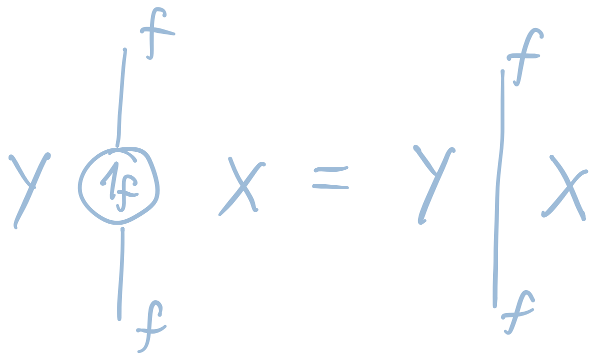
horizontal comp of 2-mor



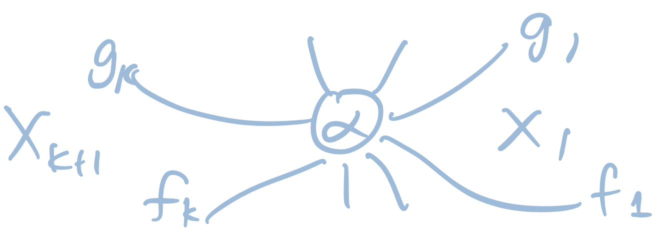


conventions





(horizontal) comp of 1-mors



$$=$$

$$\begin{tikzcd}
 & X_2 \arrow[r, "g_2"] & \cdots \arrow[dr, "g_k"] & \\
X_1 \arrow[ur, "g_1"] \arrow[dr, "f_1"'] & & & X_{k+1} \\
 & X_2' \arrow[r, "f_2"'] \arrow[u, Rightarrow, "\alpha"'] & \cdots \arrow[ur, "f_k"'] &
\end{tikzcd}$$

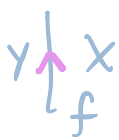
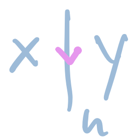
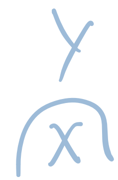
(Optional) Examples

① **Cat**
- 0 - cats
- 1 - functors
- 2 - nat. transfs

② **Bim** (weak)
- 0 - commut rings
- 1 - (ring, ring)-bimodules
  comp = $\otimes$
- 2 - bimodule homs

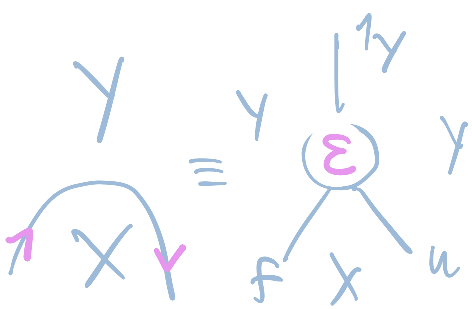
Ingredients for adjunction

Let f be left-adj to u / $f \dashv u$ / $\mathrm{Hom}(fX, Y) \cong \mathrm{Hom}(X, uY)$

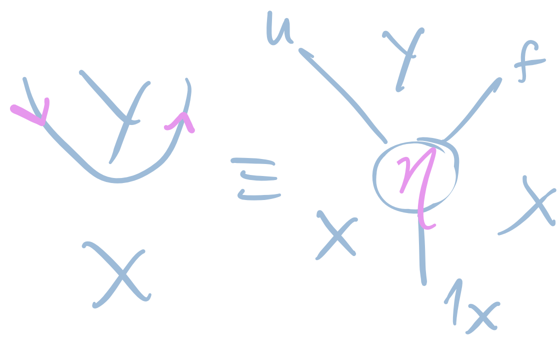
• objs X, Y • 1-mors  ,  • 2-mors $\eta =$  , $\varepsilon =$ 

counit $\eta : 1_X \Rightarrow uf$, unit $\varepsilon : fu \Rightarrow 1_Y$

unit
cap

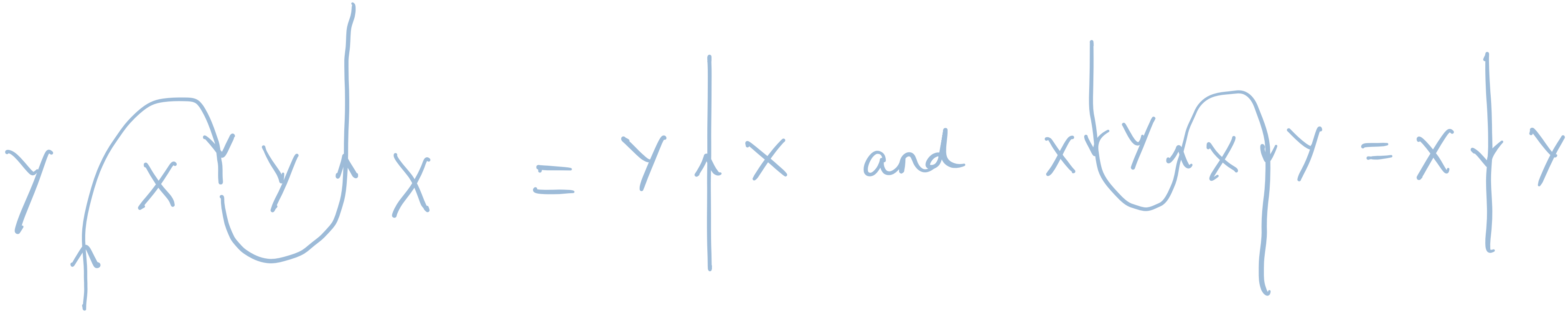


counit
cup

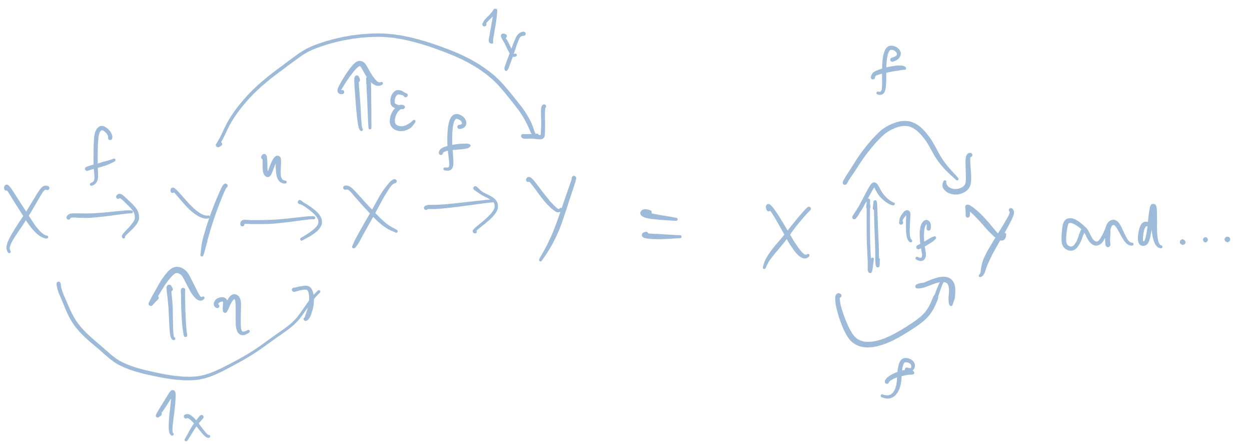


LH not'n

subject to ZIGZAG identities



In accordance w axioms of adj'n



when f is also right adjt $(u \dashv f)$ we have cup and cap diags with all possible orient'ns

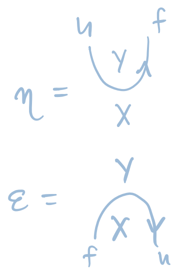
s.t. additional zigzag rels

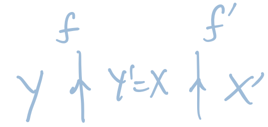
Claim/Ex the biadjt u of a given 1-mor f is unique/iso

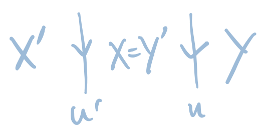
Prop $\eta, \varepsilon : f \dashv u$ and $\eta', \varepsilon' : f' \dashv u'$ be stackable adjs

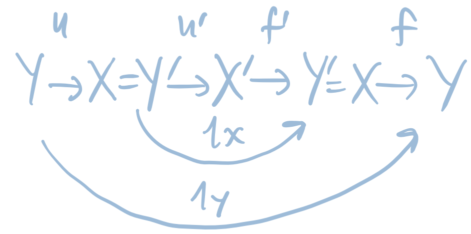
then $ff' \dashv u'u$ with $\bar{\eta} = u'\eta f' \circ \eta'$ and $\bar{\varepsilon} = \varepsilon \circ f\varepsilon' u$

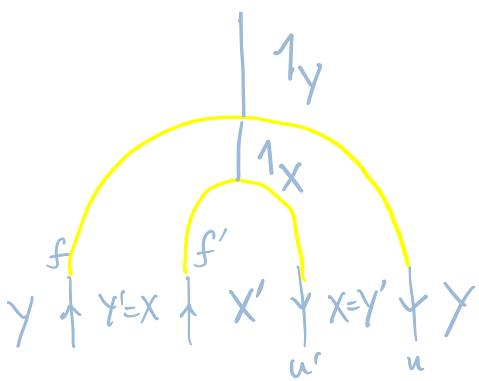
Pf

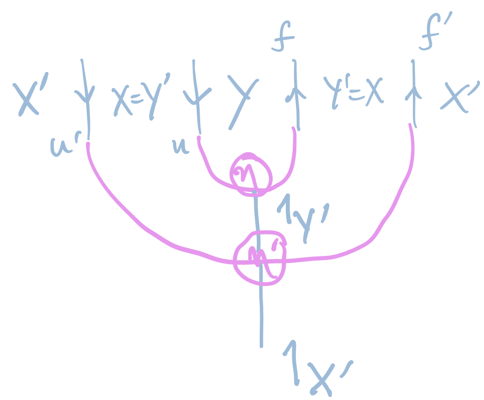


$X' \xrightarrow{f'} Y' = X \xrightarrow{f} Y =$ 

$Y \xrightarrow{u} X = Y' \xrightarrow{u'} X' =$ 



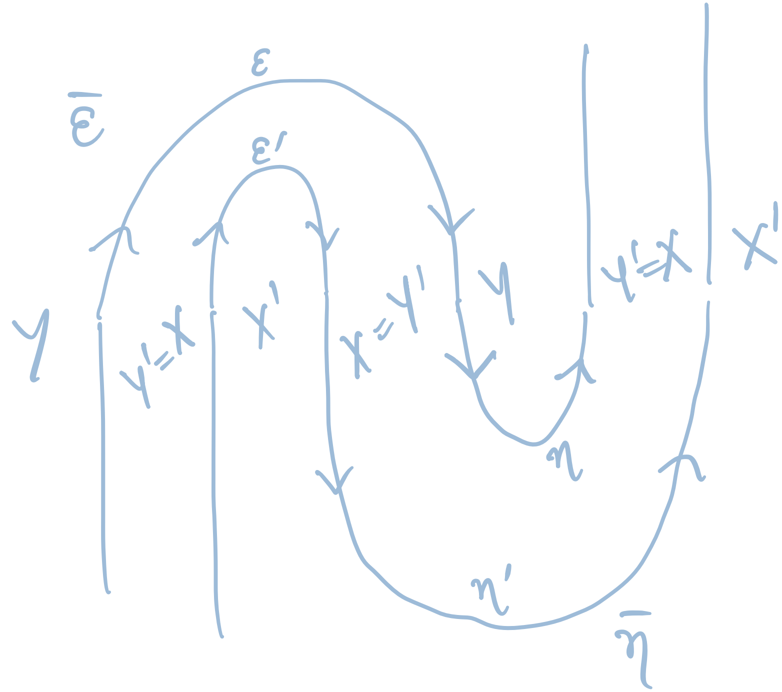
 $= \bar{\varepsilon}$

 $= \bar{\eta}$

$= ff'u'u \xRightarrow{f\varepsilon' u} \underset{1_X}{fu} \xRightarrow{\varepsilon} 1_Y$

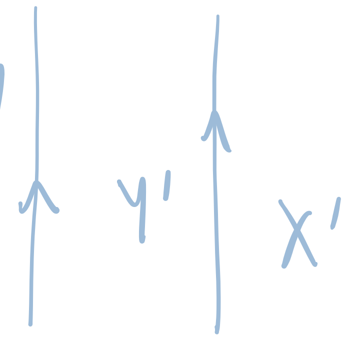
$= u'uff' \xLeftarrow{u'\eta f'} u' 1_{Y'} f' \xLeftarrow{\eta'} 1_{X'}$

Check: zig-zag identities got by straightening $\varepsilon\eta$ and $\varepsilon'\eta'$ in any order



$$
\begin{tikzcd}
X' \arrow[r] \arrow[rrrr, bend right=50, "1_{X'}"'] & Y'=X \arrow[r] \arrow[rr, bend right, "1_X"'] \arrow[rrr, phantom, shift right=6, "\eta'\Uparrow"'] \arrow[rr, phantom, shift right=2, "\eta\Uparrow"'] & Y \arrow[r] \arrow[rrrr, bend left=50, "1_Y"] \arrow[rrrr, phantom, shift left=6, "\varepsilon\Uparrow"] & X=Y' \arrow[r] \arrow[rr, bend left, "1_X"] \arrow[rr, phantom, shift left=2, "\varepsilon'\Uparrow"] & X' \arrow[r] & Y'=X \arrow[r] & Y
\end{tikzcd}
$$

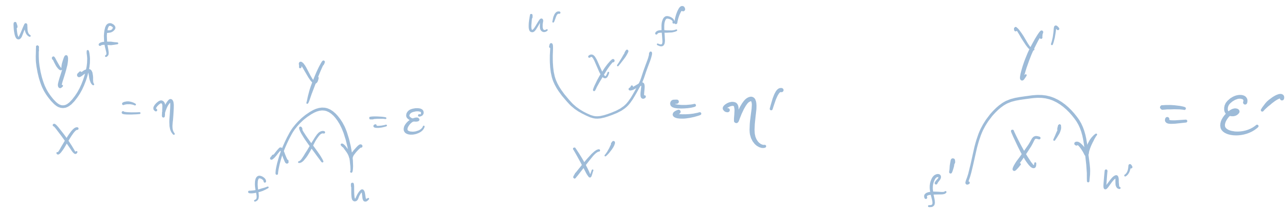
$$= X' \xrightarrow{f'} Y' \xrightarrow{f} Y$$

$=$ 

$= f f'$

Mateship under adjunction

Def given



there is a bij. M

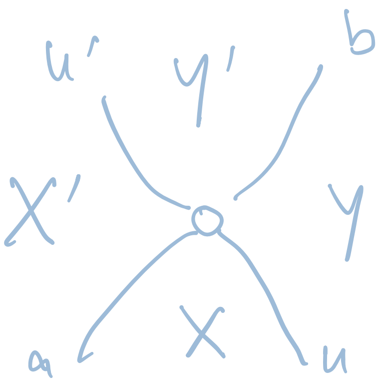
$$\{ au \Rightarrow u'b \} \xleftrightarrow{\text{mateship}} \{ f'a \Rightarrow bf \}$$

and 2-mors identified under this bij are called MATED.

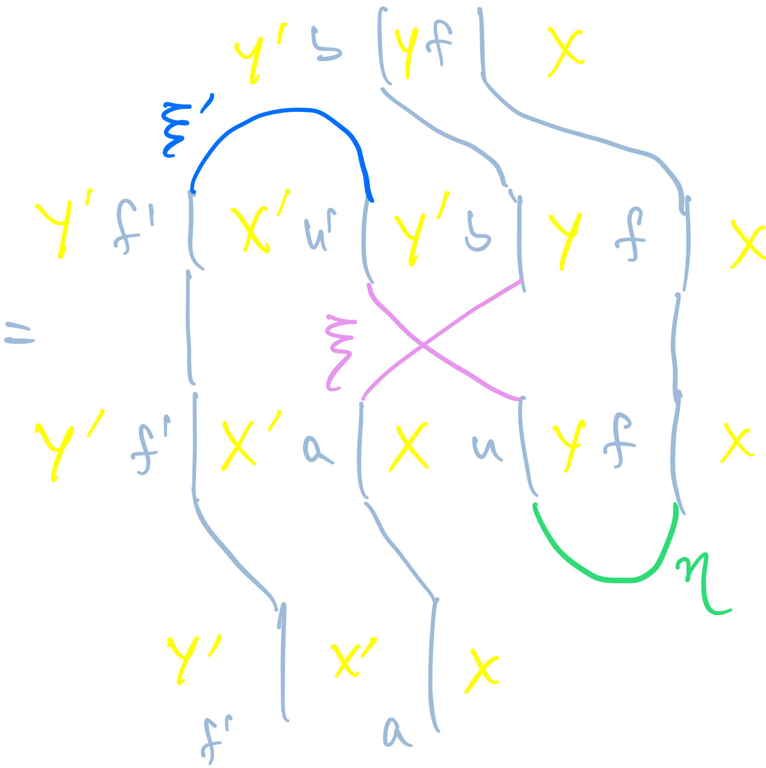
$$M(\xi) = \left(f'a \xRightarrow{f'a\eta} f'auf \xRightarrow{f'\xi f} f'u'bf \xRightarrow{\varepsilon' bf} bf \right)$$

Diagrams:

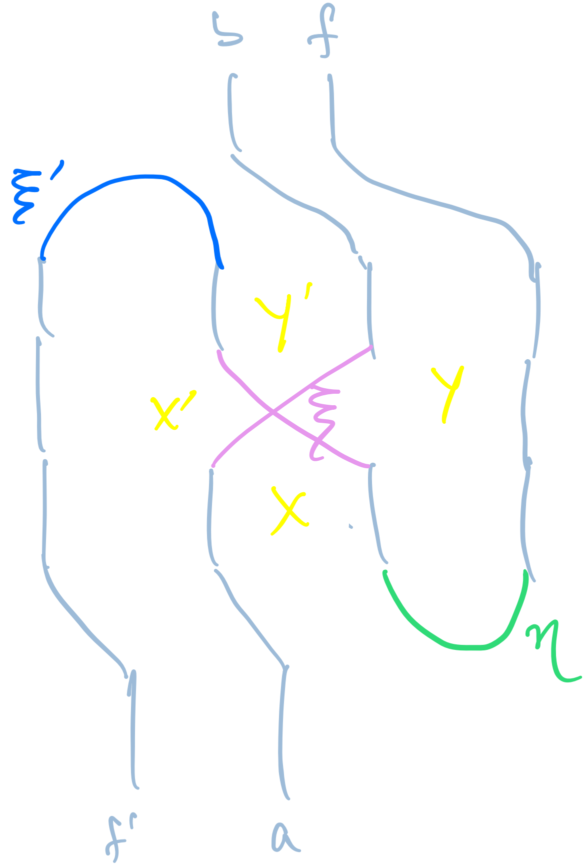
$\xi =$



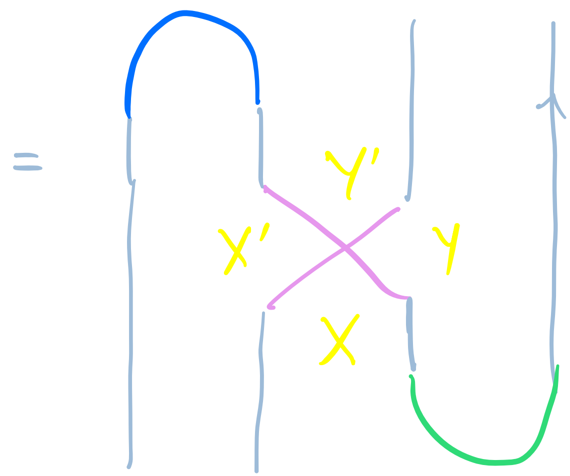
$\rightsquigarrow \mathcal{M}(\xi) =$



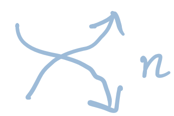
$=$

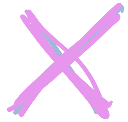
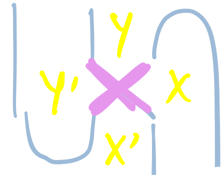


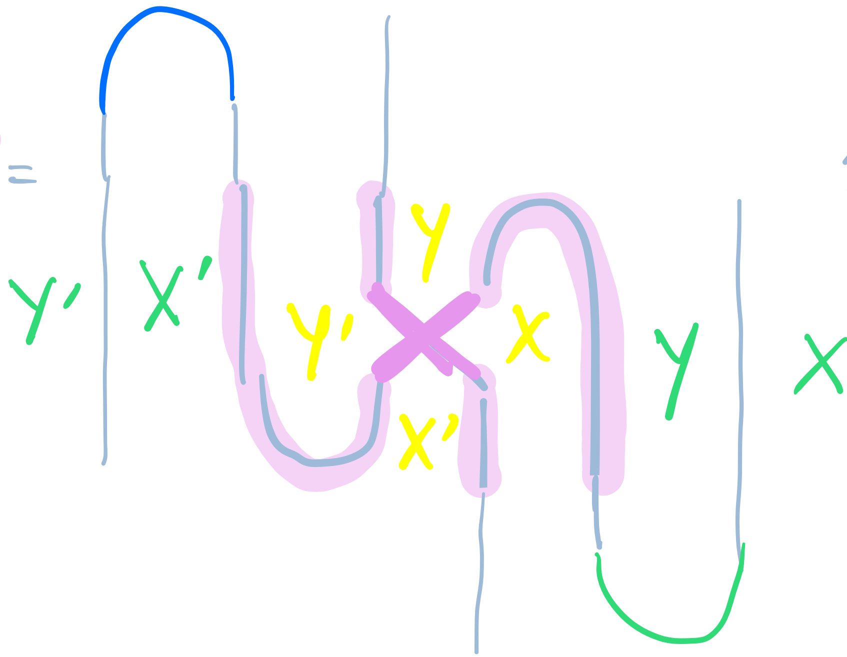
$=$

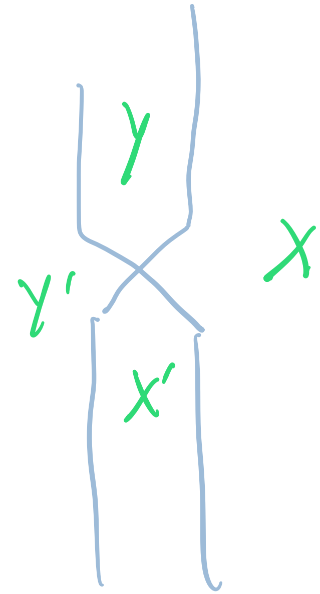


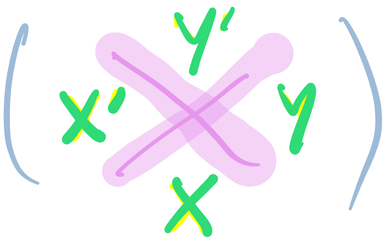
$=:$



Likewise M^{-1}() = 

Check MM^{-1}() = 

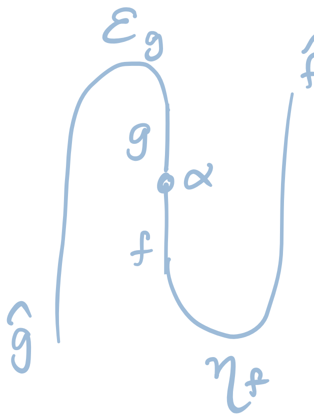
$\simeq$ 



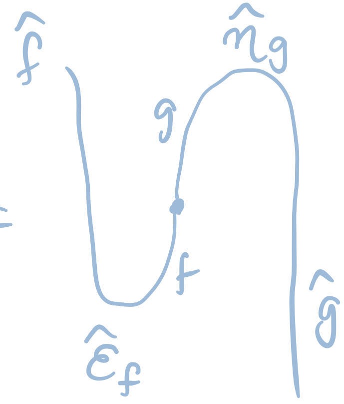
(Rem mateship satisfies all naturality axioms!)

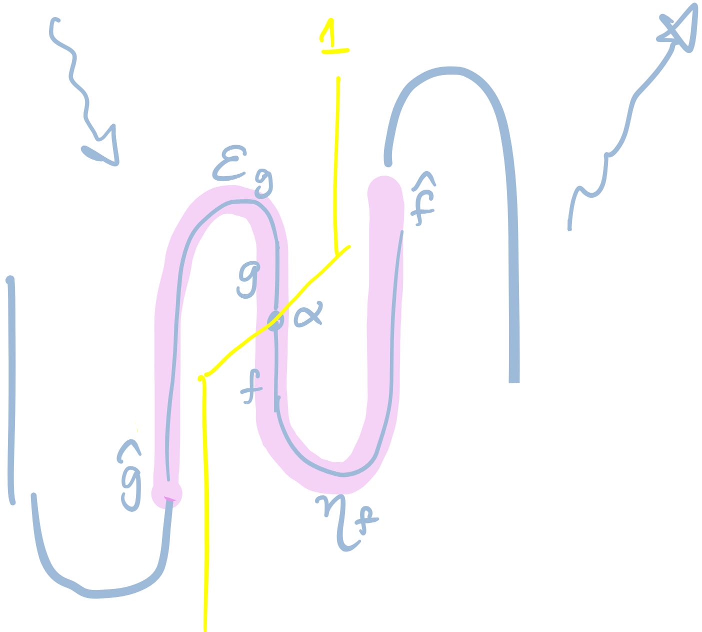
Duals for 2-mors

Given a pair of 1-mors f, g with chosen biadjts $\hat{f}, \hat{g}$ any 2-mor $f \overset{\alpha}{\Rightarrow} g$ has duals or mates: $\hat{g} \overset{{}^*\alpha,\, \alpha^*}{\Rightarrow} \hat{f}$

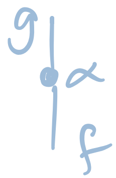
$^*\alpha :=$ 

$\overset{M}{\rightsquigarrow}$ using right adjt str

$\alpha^* :=$ 



note in general $^*\alpha \neq \alpha^*$

Def  is cyclic if $^*\alpha = \alpha^*$ for the chosen biadjt str on f, g

Notn

$a\,\underline{A}\,b := \mathrm{Hom}_{\underline{A}}(a,b) \ni x,y \Rightarrow \mathrm{Hom}(x,y) =: A(x,y)$

$\Rightarrow$ if x,y do not share src & target

3. THE 2-CATEGORY $\mathcal{U}$

$$\mathcal{U}^*(x,y) := \bigoplus_{s\in\mathbb{Z}} \mathcal{U}(x\{s\},y)$$

Want #1: ENRICHED HOM of a 2-cat

use $\mathcal{U}^*$ for "deg" of 2-hom, and gr ab gp of pair of 1-mors

Warning: $\mathcal{U}^*$ does not have the right Grothendieck group

Solution: $\mathcal{U} \subseteq \mathcal{U}^*$ got by res to deg-preserving 2-mors

- $x \not\cong x\{s\}$ since the shifting id map is not deg-pr
- homs not gr ab gps but deg 0 mors form an ab gp

also, gr ab gps are naturally assoc. to homs hom = 1-mor

by $\mathcal{U} \to \mathcal{U}^*$

Want #2: $K_0(\mathcal{U})$

Use primitive orthogonal idempotents $\{e_i\}_{i=1}^k$ constructed in $\mathcal{U}(x,x)$

to decompose 1_x if idems split, then $x = \bigoplus x_i$ where $x_i = \mathrm{Im}(e_i)$ and $\mathrm{Hom}(e_i,e_j) = e_i\mathcal{U}(x,x)e_j$

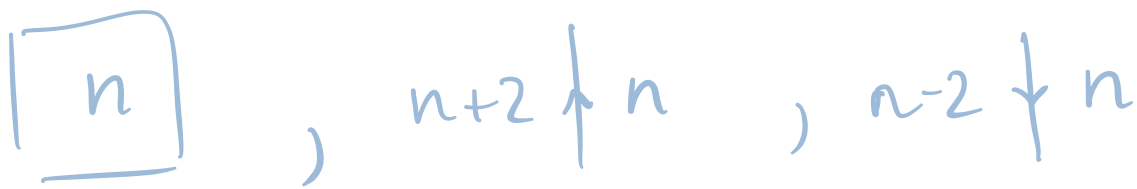
Use idempotent completion for $\oplus$ decomp $\quad Kar(\mathcal{U}) = \dot{\mathcal{U}}$

Outcome: $\dot{\mathcal{U}}$ categorifies $\dot{U}$

The 2-CAT $\mathcal{U}^*$

0 – $n \in \mathbb{Z}$

1 – formal direct sums of composites of



$(n \in \mathbb{Z})$

$\mathbb{1}_n\{s\}$, $\quad \mathbb{1}_{n+2}\mathcal{E}\mathbb{1}_n\{s\}$, $\quad \mathbb{1}_{n-2}\mathcal{F}\mathbb{1}_n\{s\}$ $\quad (s \in \mathbb{Z})$

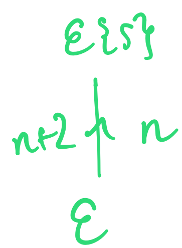
$\rightsquigarrow \mathcal{U}^*(n,m)$:

gen obj: $\mathbb{1}_m \mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1} \cdots \mathcal{E}^{\alpha_k}\mathcal{F}^{\beta_k}\mathbb{1}_n\{s\}$ $\qquad m = n + 2\Sigma(\alpha_i - \beta_i)$

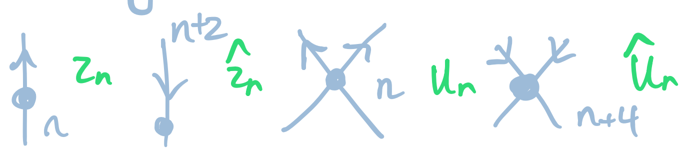
mors: $x, y \in \mathcal{U}^*$ $\quad \mathcal{U}^*(x,y)$: deg 0 2-mors $1_x : x \Rightarrow x$

deg $\pm s$ 2-mors : $x \Rightarrow x\{s\} \Rightarrow x$

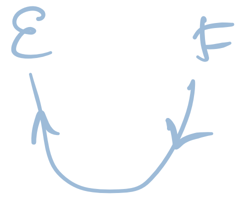
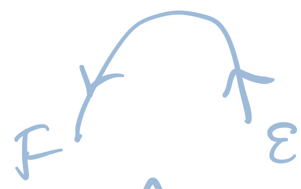
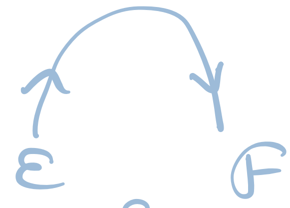
eg



deg ± 2 mors :



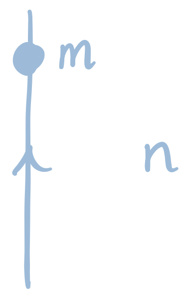
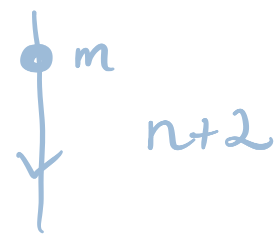
Plus

			
η_n	$\hat{\varepsilon}_n$	$\hat{\eta}_n$	ε_n
deg $n+1$	$1-n$	$n+1$	$1-n$

Composition:

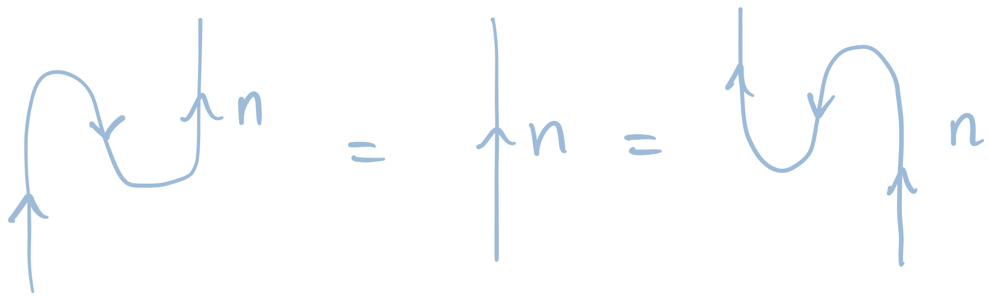
$$\mathcal{U}^*(n,n') \times \mathcal{U}^*(n',n'') \longrightarrow \mathcal{U}^*(n,n'')$$

is juxtaposition

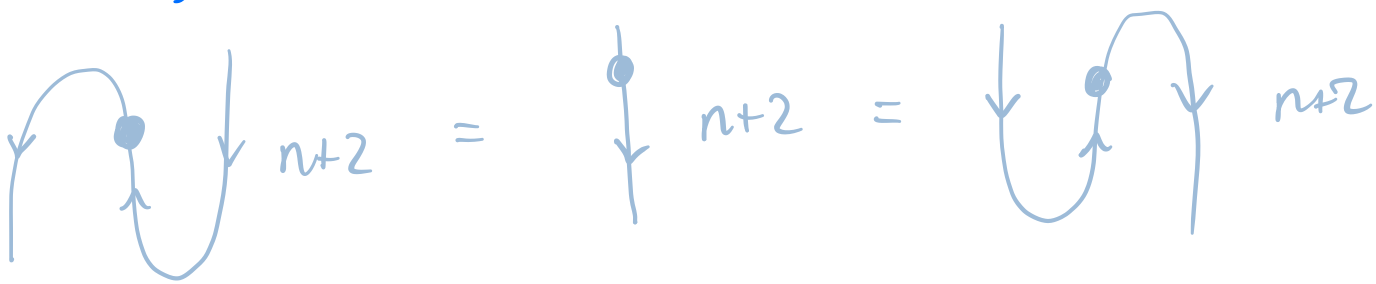
Relations! Let $z_n^m =$  $\hat{z}_n^m =$ 

Biadjointness

There are unit, counit pairs $(\eta_n, \varepsilon_{n+2})$, $(\hat{\varepsilon}_n, \hat{\eta}_{n-2})$ and biadjoints $\mathbb{1}_{n+2}\mathcal{E}\mathbb{1}_n \dashv \mathbb{1}_n\mathcal{F}\mathbb{1}_{n+2} \dashv \mathbb{1}_{n+2}\mathcal{E}\mathbb{1}_n$

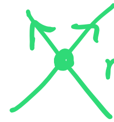


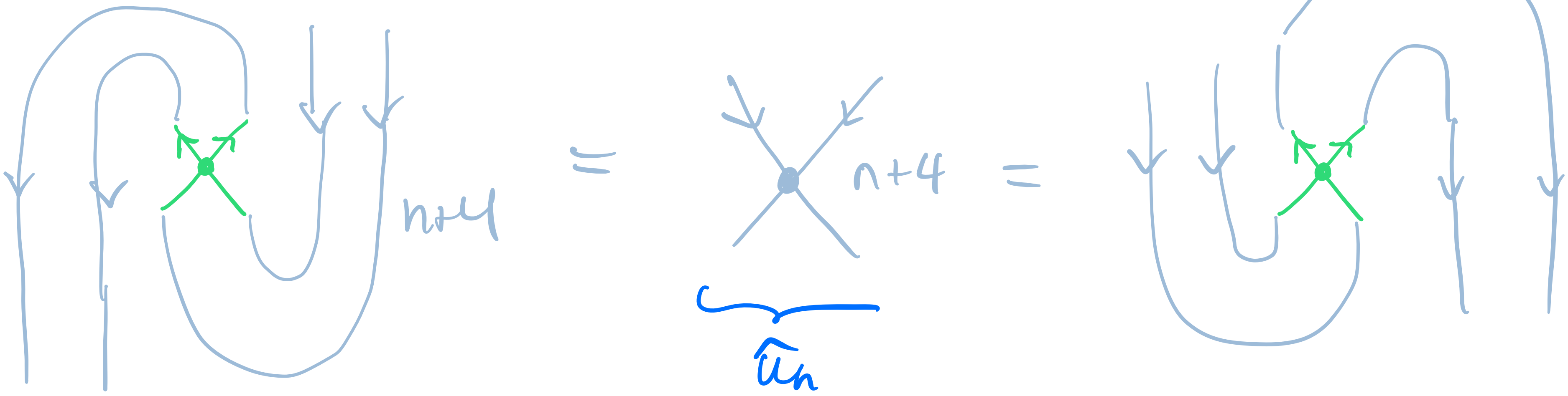
Duality for $z_n, \hat{z}_n$



Duality for $u_n, \hat{u}_n$

${}^*u_n = u_n^*$

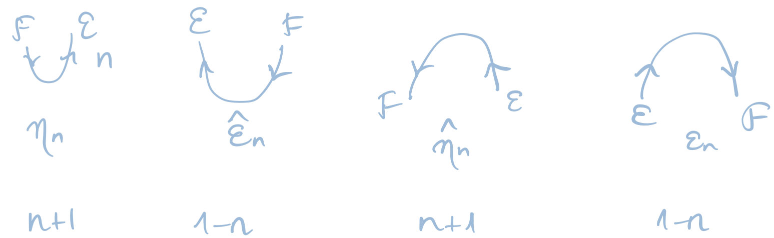


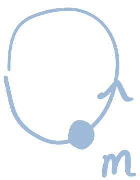


is thus 2-sided dual to u_n

The three axioms above imply that all the morphisms in $\mathcal{U}^*$ are cyclic 2-morphisms with respect to the biadjoint structure each 1-morphism inherits from the definitions above. Hence, these axioms ensure that topological deformations of a diagram that preserve the boundary result in a diagram representing the same 2-morphism.

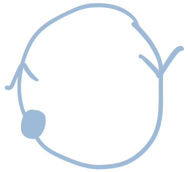
Positive degree of closed bubbles





$= \hat{\eta}_n z_n^m \eta_n \qquad 2(n+1) + 2m = 2(n+m+1)$

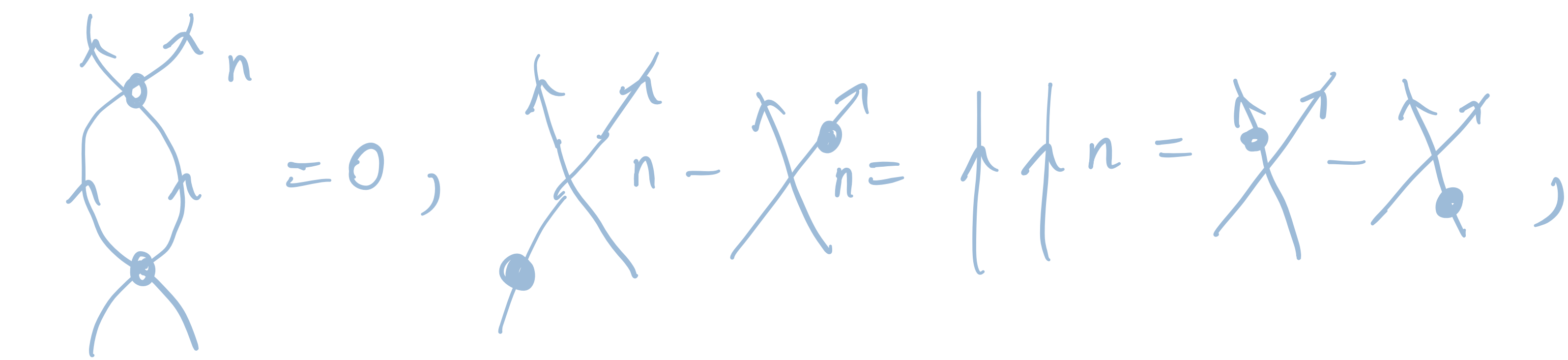
$= 0 \quad$ if $\quad m < -n-1$

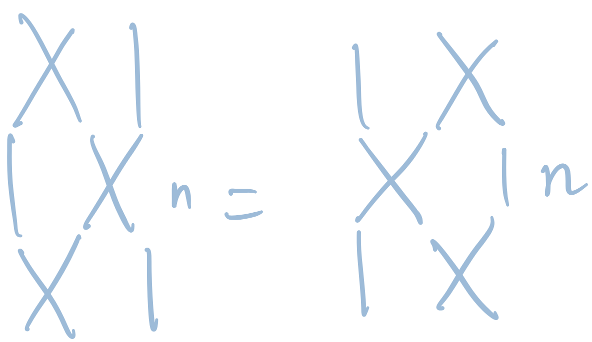

$= \mathcal{E}_n \hat{\Sigma}^m_n \hat{\mathcal{E}}_n \qquad (1-n)+2m+(1-n) = 2(1-n+m)$

$= 0$ if $m < n-1$

Declare bubbles of negative degree $= 0$. (nonobvious conseq: any closed diagram of negative deg $= 0$)

NilHecke action:



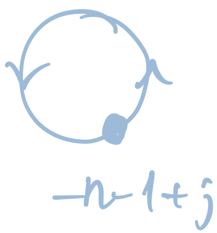


These rels $\Rightarrow$ $NH_a \circlearrowright \mathcal{U}^*(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$ $\forall n \in \mathbb{Z}$.

$$x_i \mapsto z_{n+i}$$

$$u_j \mapsto u_{n+j}$$

For the final rels need **FAKE BUBBLES**



$n \geq 0$

$n \leq 0$

$0 \leq j \leq n$ $\qquad$ $0 \leq l \leq -n$

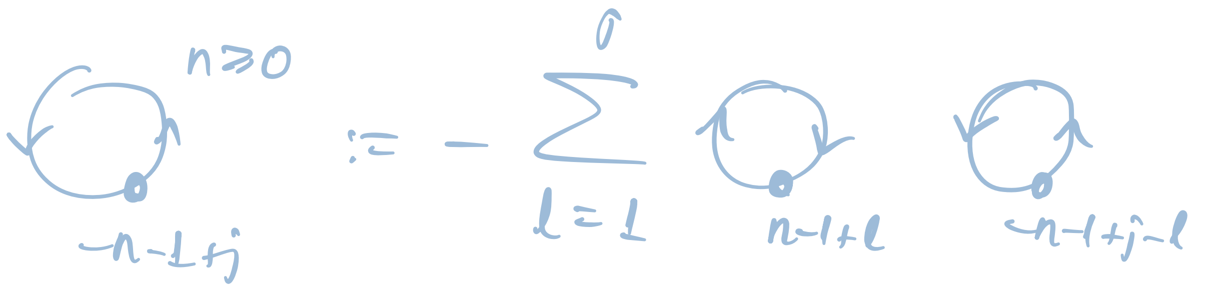
degrees are ok! $\quad 2(1+n-n-1+j = j) \geq 0$ $\qquad$ $2(1-n+n-1+l = l) \geq 0$

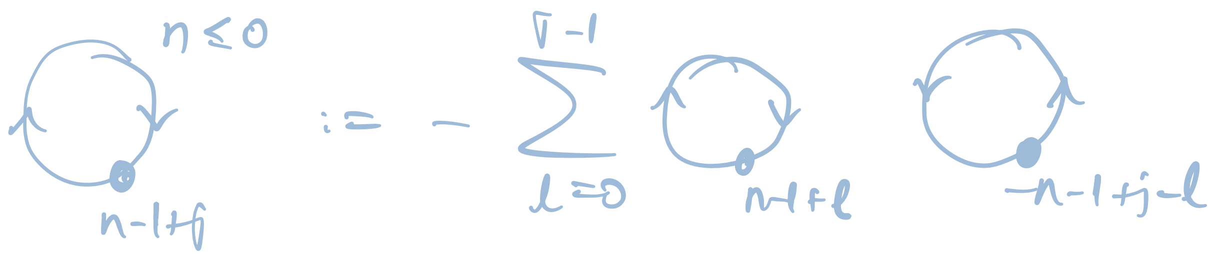
But pictures do not make sense: $-n-1+j < 0$ mult. of $\bullet$

To make more sense of these, declare

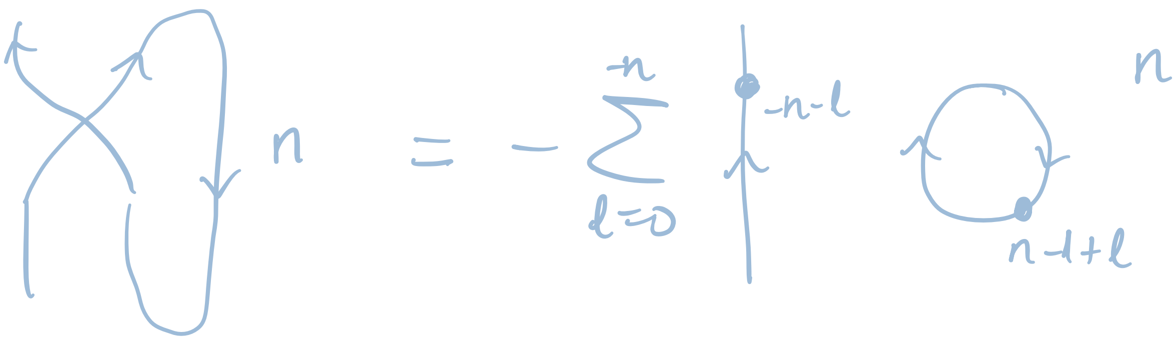
init:

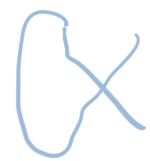
iterate: $1 \leq j \leq n$





Reduction to bubbles



and sim. for 

Looking ahead: Choice of 2-mors in categorification will depend on semilinear form

Eg $\langle E^a 1_n, E^a 1_n \rangle = \mathrm{gr\,dim}\, H^*(Gr(a,\infty)) = \prod^{a} \frac{1}{1-q^{2i}}$

$\Rightarrow \langle E 1_n, E 1_n \rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + \cdots$

Use the form to guess generating 2-mors

$\deg\left(\uparrow_n \right) = 0$ contributes $q^0 = 1$ to grdim

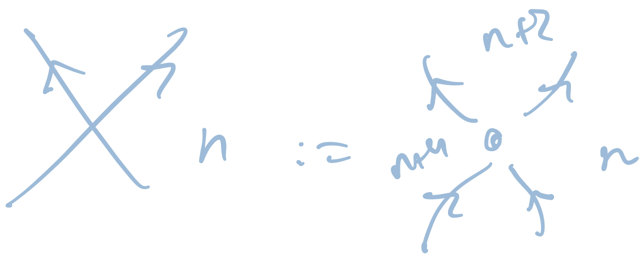
$\deg\left(\overset{\uparrow}{\bullet} \right) = 2$ $\quad q^2 \quad \ldots$

$\mathrm{gdim}\, \mathrm{Hom}_{\dot{\mathcal{U}}}(\mathcal{E}1_n, \mathcal{E}1_n) = \deg(\uparrow) + \deg\left(\overset{\uparrow}{\bullet}\right) + \deg\left(\overset{\uparrow}{\bullet_2}\right) + \cdots$

$= 1 + q^2 + q^4 + \cdots$

Eg $\quad \langle E^2 1_n, E^2 1_n \rangle = [2][2]\frac{1}{1-q^2}\frac{1}{1-q^4} = \underbrace{(1-q^{-2})}\left(\frac{1}{1-q^2}\right)$

need deg -2
gen mor

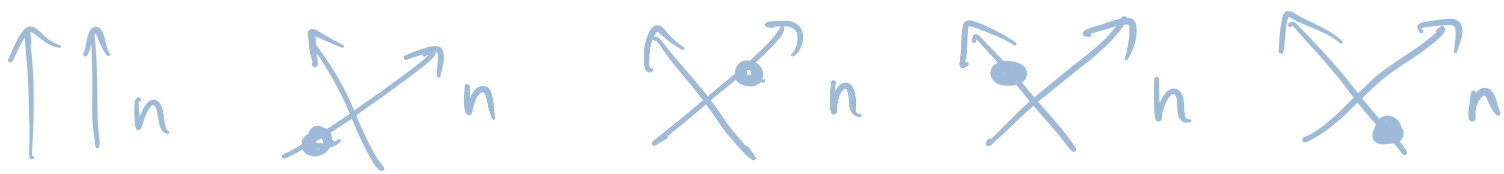


to the rescue!

$$\mathrm{gdim}\, \dot{\mathcal{U}}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n) = (1-q^{-2})(1+q^2+q^4+\cdots)^2$$

deg() $= -4 \Rightarrow$  $= 0$

& the space of deg 0 2-mors is 3-dim, spanned by

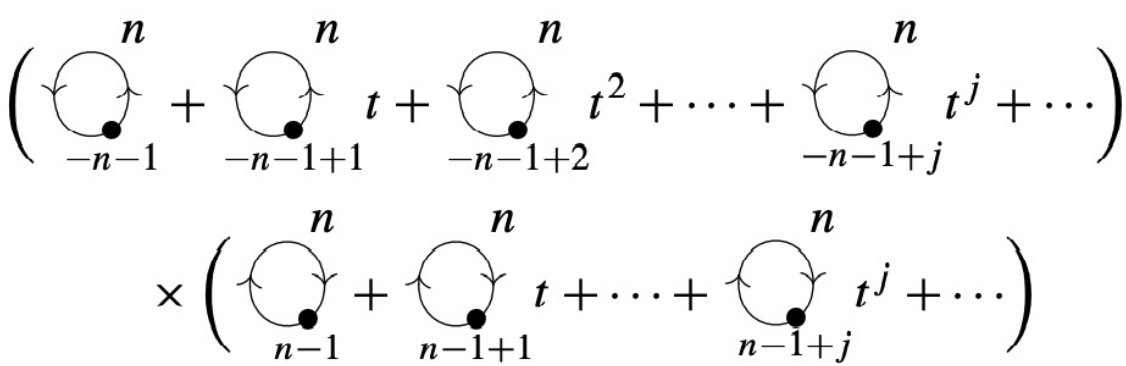


not lin. indep!

See §5.4 Lau10 for more helpful rels

prop 5.2, 5.3, 5.4

Proposition 5.5 *(Infinite Grassmannian relations). The following product:*



is equal to Id_{1_n} *for all* n*, where* t *is a formal variable. This is in analogy with the generators of the cohomology ring* $H^*(Gr(n,\infty))$ *of the infinite Grassmannian (see Section 6):*

$$(1 + x_1 t + x_2 t^2 + \cdots + x_j t^j + \cdots)(1 + y_1 t + y_2 t^2 + \cdots + y_j t^j + \cdots) = 1.$$

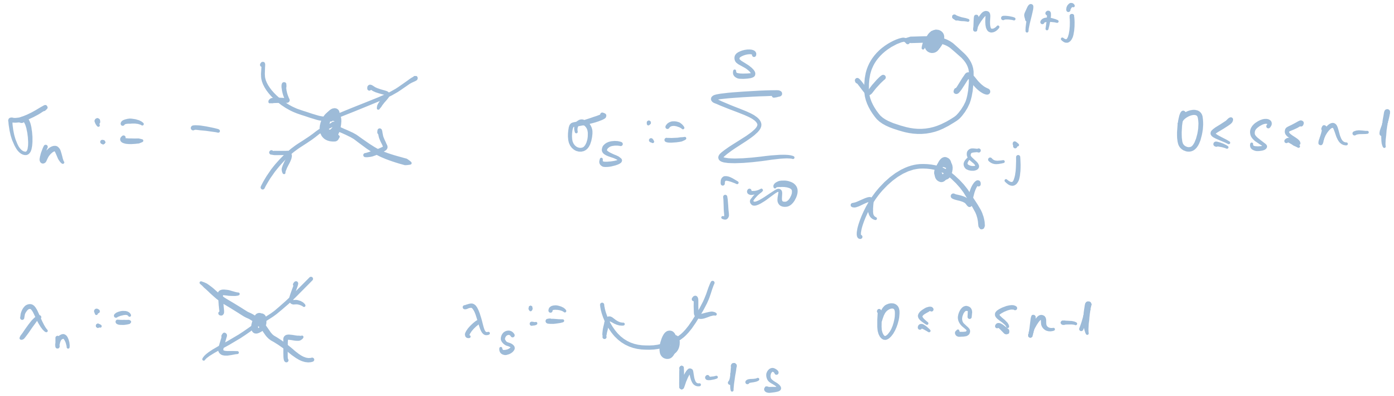
4. $\mathcal{U}$ categorifies sl_2 relations

Thm (L10 5.10) There are decompositions

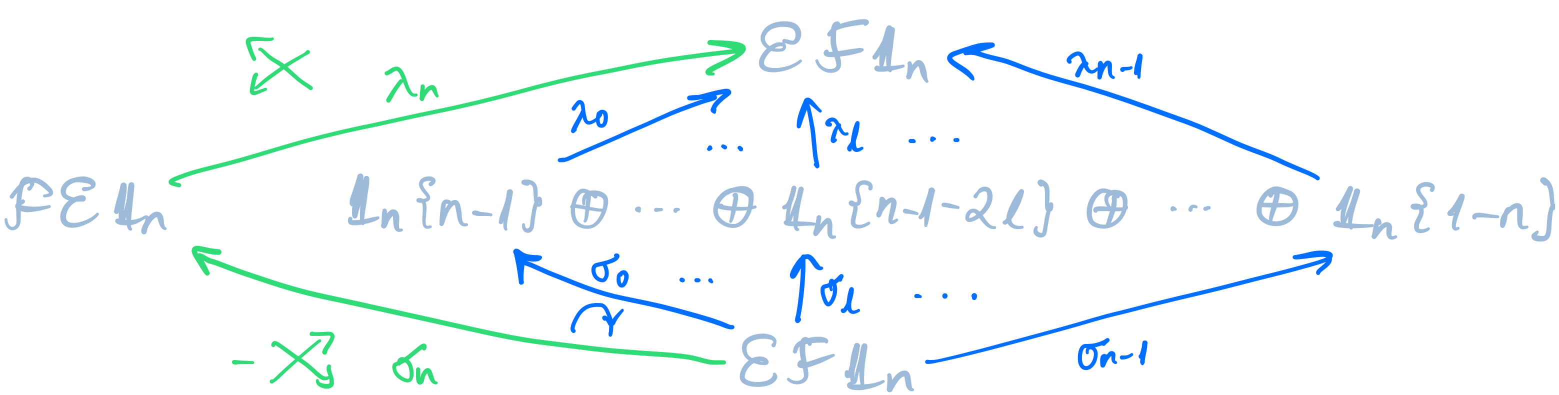
$$\mathcal{E}\mathcal{F}\mathbb{1}_n \cong \mathcal{F}\mathcal{E}\mathbb{1}_n \oplus_{[n]} \mathbb{1}_n \qquad n \geq 0$$

$$\mathcal{F}\mathcal{E}\mathbb{1}_n \cong \mathcal{E}\mathcal{F}\mathbb{1}_n \oplus_{[-n]} \mathbb{1}_n \qquad n \leq 0$$

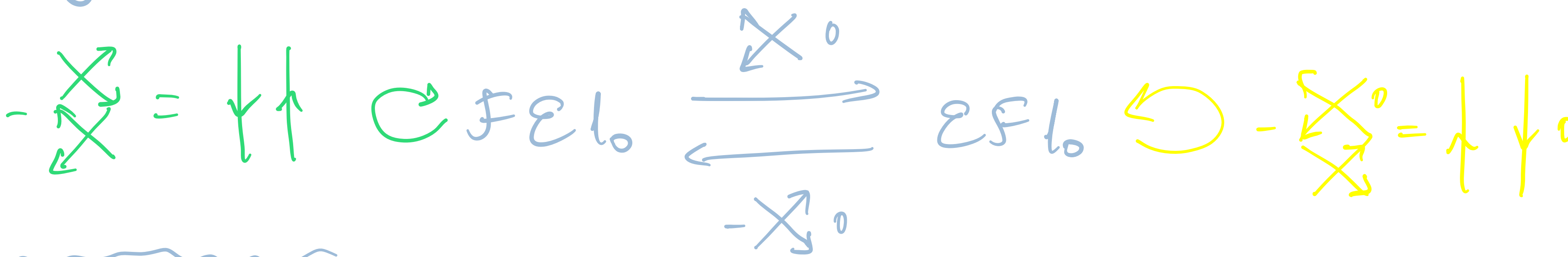
Pf Declare



This system of 2-mors provides the decomp $\mathcal{E}\mathcal{F}\mathbb{1}_n \cong \mathcal{F}\mathcal{E}\mathbb{1}_n \oplus_{[n]} \mathbb{1}_n$ $(n \geq 0)$



Eg ($n=0$, $\mathcal{E}\mathcal{F}1_0 \cong \mathcal{F}\mathcal{E}1_0$)



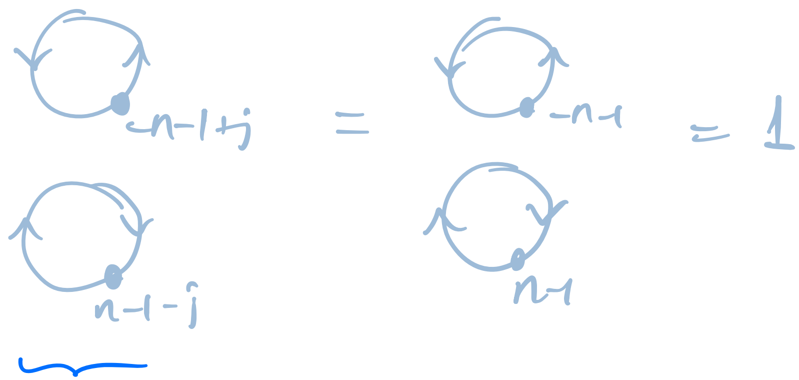
With the shifts the maps $\lambda_s \sigma_s =: e_s$ are degree 0

Claim: $\{e_s : 0 \le s \le -n\}$ is a collection of orthogonal idempotents decomposing $\mathrm{Id}_{\mathcal{E}\mathcal{F}1_n}$

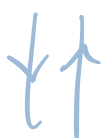
$$(\lambda_s \sigma_s)^2 = \lambda_s \underbrace{\sigma_s \lambda_s}_{1} \sigma_s$$

when $0 \le s \le n-1$

$$\sigma_s \lambda_s = \sum_{j=0}^{1}$$



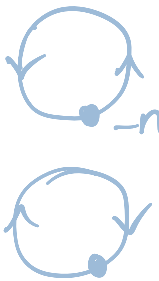
$\sigma_n \lambda_n = \mathrm{Id}_{\mathcal{F}\mathcal{E}1_n} =$



(rel 5.19 in Lau10)

Orthogonality: $e_s e_{s'} = \lambda_s \underbrace{\sigma_s \lambda_{s'}} \sigma_{s'}$ $\qquad 0 \le s' < s \le n-1$

$\sum_{j=0}^{s}$



$= \sum_{j=0}^{s-s'}$

5.25

$= 0$

deg < 0 if $j > s-s'$

Similarly for $s < s'$ we get $\sigma_s \lambda_{s'} = 0$

Need another argument for $\sigma_s \lambda_n \qquad s < n$

And finally show

$$\sum_{s=0}^{n} e_s = \mathrm{Id}_{EF\mathbb{1}_n} \qquad (5.18) \qquad \#$$

also §7 of LW

5. The 2-rep Γ_N An action on partial flag varieties

Idea: in search of "rels" in $\mathcal{U}$ what should hold consider cat'l actions of $\dot{U}$ and nat transfs of functors coming from gen'g 2-mors

§3.4.1 "A model ex of a cat $\dot{U}$ action"

Let V^N or V_N denote the $(N+1)$-dim irrep of $\dot{U}$

V^N can be constructed using cats of gr mods over coho of Gr's

For $0 \leq k \leq N$ denote by $Gr(k,N)$ the cx proj var of k-planes in $\mathbb{C}^N$

Fact: $\underbrace{H^{\cdot}(Gr(k,N), \mathbb{Q})}_{=:H_k} = \bigoplus_{0 \leq i \leq k(N-k)} H^i(G_k, \mathbb{Q})$ G_k: $Gr(k,N)$ i.e. this coho ring is a graded $\mathbb{Q}$-alg

and

$$H_k = \mathbb{Q}[c_1 \ldots c_k, \bar{c}_1 \ldots \bar{c}_{N-k}]/I_{k,N} \qquad \deg c_i = \deg \bar{c}_i = 2i$$

$$I_{k,N} \leftarrow (1 + c_1 t + \cdots + c_k t^k)(1 + \bar{c}_1 t + \cdots + \bar{c}_{N-k} t^{N-k}) = 1$$

Example $N=5, \quad k=2$

$$H_2 = \mathbb{Q}[c_1, c_2, \bar{c}_1, \bar{c}_2, \bar{c}_3] / (c_1+\bar{c}_1,\ c_1\bar{c}_1 + c_2 + \bar{c}_2,\ c_2\bar{c}_1 + c_1\bar{c}_2 + \bar{c}_3,\ c_1\bar{c}_3 + c_2\bar{c}_2,\ c_2\bar{c}_3)$$

$$1 = (1+c_1 t + c_2 t^2)(1 + \bar{c}_1 t + \bar{c}_2 t^2 + \bar{c}_3 t^3)$$

$$= 1 + (c_1 + \bar{c}_1)t + (c_1\bar{c}_1 + c_2 + \bar{c}_2)t^2 + (c_2\bar{c}_1 + c_1\bar{c}_2 + \bar{c}_3)t^3$$
$$+ (c_1\bar{c}_3 + c_2\bar{c}_2)t^4 + c_2\bar{c}_3 t^5$$

$$\Rightarrow \quad \bar{c}_1 = -c_1$$

$$\bar{c}_2 = c_1^2 - c_2$$

$$\bar{c}_3 = c_2 c_1 - c_1(c_1^2 - c_2) = 2c_1c_2 - c_1^3 + \cancel{c_1 c_2}$$

$$\therefore \quad H_2 = \mathbb{Q}[c_1, c_2] / (c_1^4 - 3c_1^2 c_2 + c_2^2,\ 2c_1c_2^2 - c_1^3 c_2)$$

In general, ~~using~~ modulo $I_{k,N}$, can solve for $\bar{c}_i$ in terms of c_j

if $k \le N-k$ $(N-2k \geq 0)$

Assume wlog $N-2k \geq 0$ and write $\bar{c}_i$ in terms of c_j

Fact: remaining k rels on c_1 — c_k are given by the first column of

$$\begin{bmatrix} c_1 & 1 & 0 & \cdots & 0 \\ -c_2 & 0 & 1 & & \vdots \\ c_3 & 0 & & \ddots & \\ \vdots & \vdots & & & 1 \\ (-1)^{k+1}c_k & 0 & \cdots & & 0 \end{bmatrix}^{N-k+1}$$

Ref Hil82 p107

Eg/Ex. confirm in prev ex

$$\begin{bmatrix} c_1 & 1 \\ -c_2 & 0 \end{bmatrix}^{\cancel{5-2+1}\ 4} = \begin{bmatrix} c_1^4 - 3c_1^2c_2 + c_2^2 & * \\ -c_1^3c_2 + 2c_1c_2^2 & * \end{bmatrix}$$

Let H_k-pmod denote the cat of f.g. proj H_k-modules

Let $\mathcal{V}^N_n = H_k\text{-pmod}$ $(n = 2k-N)$

Fact H_k are gr loc rings. Consequence $K_0(\mathcal{V}^N_n)$ is a free $\mathbb{Z}[q, q^{-1}]$

mod gen'd by unique indec proj

Cf aaron1: $F \circlearrowright \mathcal{V}_N := \bigoplus H_k\text{-gmod}$

Cor $K_0(\mathcal{V}_n^N) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Q}(q) = \mathbb{Q}(q)$

Cor $\mathcal{V}^N := \bigoplus_{n=-N+2k}^{N} \mathcal{V}_n^N$ categorifies V^N

$$K_0(\mathcal{V}^N) = \bigoplus K_0(\mathcal{V}_n^N) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Q}(q) \cong V^N$$

as $\mathbb{Q}(q)$-v.s.

Qu: How do $E1_n$, $F1_n$ act?

An: By "correspondences", i.e. let

$$Fl(k, k+1, N) := \{(0 \subseteq W_k \subseteq W_{k+1} \subseteq \mathbb{C}^N) : \dim W_i = i\}$$

$Fl(k,k+1,N) \xrightarrow{\text{send pair to } W_k} Gr(k,N)$

$Fl(k,k+1,N) \xrightarrow{\text{send pair to } W_{k+1}} Gr(k+1,N)$

These "forgetful" maps give rise to inds on coho.

Let $H_{k,k+1} := H^*(Fl(k,k+1,N), \mathbb{Q})$ for 18r!

$$x^k$$

$$a \in H_k \nearrow \qquad \nwarrow H_{k+1} \ni b$$

$b \in H_{k+1} \subset V \ni v$

$H_{k,k+1} \otimes_{H_k} V \circlearrowleft H_k$

These coho maps endow $H_{k,k+1}$ with the str of a

(H_{k+1}, H_k)-bim $= (H_k, H_{k+1})$-bim

$x \mapsto bxa = axb = abx$?

Fact: all these coho rings are commut!

Upshot: get functors btw cats by $\otimes$ w bims

In particular, $E1_n$ and 1_nF ? act by $H_{k+1,k}$ and $H_{k,k+1}$ resp, w $n = 2k-N$.

More precisely, $\mathbb{1}_n := H_k \otimes_{H_k} - \; : H_k\text{-pmod} \to H_k\text{-pmod}$

$\mathcal{E}\mathbb{1}_n := H_{k+1,k} \otimes_{H_k \{1-N+k\}} - \; : H_k\text{-pmod} \to H_{k+1}\text{-pmod}$

order dont matter?

$\mathcal{F}\mathbb{1}_{n+2} := H_{k,k+1} \otimes_{H_{k+1}\{-k\}} - \; : H_{k+1}\text{-pmod} \to H_k\text{-pmod}$

Note gr shifts needed to ensure qu $\mathfrak{sl}_2$ rels

$$\mathcal{E}\mathcal{F}\mathbb{1}_n \cong \mathcal{F}\mathcal{E}\mathbb{1}_n \oplus \mathbb{1}_n^{\oplus[n]} \qquad n \geq 0$$

$$\mathcal{F}\mathcal{E}\mathbb{1}_n \cong \mathcal{E}\mathcal{F}\mathbb{1}_n \oplus \mathbb{1}_n^{\oplus[-n]} \qquad n \leq 0$$

where $\mathbb{1}_n^{\oplus[n]} := \mathbb{1}_n\{n-1\} \oplus \mathbb{1}_n\{n-3\} \oplus \cdots \oplus \mathbb{1}_n\{1-n\}$

Fact: Functors ☺ have L/R adjs and commute w gr shift on gr mods

$$\begin{tikzcd}
\mathcal{V}^N_{-N} \arrow[r, bend left, "\mathcal{E}\mathbf{1}_{-N}"] & \arrow[l, bend left, "\mathcal{F}\mathbf{1}_N"] \cdots \mathcal{V}^N_{n-2} \arrow[r, bend left, "\mathcal{E}\mathbf{1}_{n-2}"] & \mathcal{V}^N_n \arrow[l, bend left, "\mathcal{F}\mathbf{1}_n"] \arrow[r, bend left, "\mathcal{E}\mathbf{1}_n"] & \mathcal{V}^N_{n+2} \arrow[l, bend left, "\mathcal{F}\mathbf{1}_{n+2}"] \cdots \arrow[r, bend left, "\mathcal{E}\mathbf{1}_N"] & \mathcal{V}^N_N \arrow[l, bend left, "\mathcal{F}\mathbf{1}_N"]
\end{tikzcd}$$

$$\begin{tikzcd}
H_0 \arrow[r, bend left, "H_{1,0}"] & \arrow[l, bend left, "H_{0,1}"] \cdots H_{k-1} \arrow[r, bend left, "H_{k,k-1}"] & H_k \arrow[l, bend left, "H_{k-1,k}"] \arrow[r, bend left, "H_{k+1,k}"] & H_{k+1} \arrow[l, bend left, "H_{k,k+1}"] \cdots \arrow[r, bend left, "H_{N,N-1}"] & H_N \arrow[l, bend left, "H_{N-1,N}"]
\end{tikzcd}$$

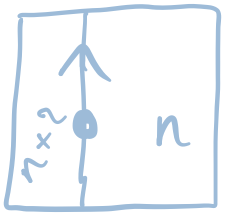
FIGURE 4. The top diagram illustrates the weight space decomposition of categories and functors appearing in the categorification of V^N. The lower diagram illustrates the graded rings whose module categories give weight space categories and the bimodules giving rise to functors on the module categories.

§ 3.4.2 Natural transformations

- $H_{k,k+1} := \mathbb{Q}[c_1 \ldots c_k \,;\, \xi \,;\, \bar{c}_1 \ldots \bar{c}_{N-k-1}] / I_{k,k+1,N}$

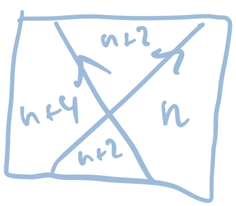
$$(1+c_1t+\ldots+c_kt^k)(1+\xi t)(1+\bar{c}_1t+\ldots+\bar{c}_{N-k-1}t^{N-k-1}) = 1$$

where ξ has deg 2 and is the chern class of the line bundle whose fibre over (W_k, W_{k+1}) is the line W_{k+1}/W_k

- 

: $H_{k+1,k} \otimes_{H_k} - \xRightarrow{\text{mult by } \xi} H_{k+1,k} \otimes_{H_k} -$

deg 2 nat transf got by mult by ξ

- 

: $H_{k,k+1,k+2} \otimes_{H_k} - \xRightarrow{\partial_1} H_{k,k+1,k+2} \otimes_{H_k} -$

deg-2 nat transf got by applying divided diff in ξ_1, ξ_2

push $\searrow$ $H_{k,k+2}$ $\nearrow$ pull

the bim corresp to $\mathcal{E}\mathcal{E}1_n$ is $H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k}$

$= H^*(Fl(k,k+1,k+2,N),\mathbb{Q})$

more explicitly, we that $H_{k\,k+1\,k+2} = \mathbb{Q}[c_1 - c_k\ \xi_1\ \xi_2\ \bar{c}_1 - \bar{c}_{N-k-2}] \big/ I_{k\,k+1\,k+2}$

$$\rightsquigarrow I_{k\,k+1\,k+2}$$

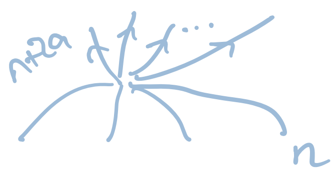
$$(1 + c_1 t + \dots + c_k t^k)(1+\xi_1 t)(1+\xi_2 t)(1 + \bar{c}_1 t + \dots + \bar{c}_{N-k-2} t^{N-k-2})$$

again $\xi_i = c_1\left(\frac{W_{k+i}}{W_{k+i-1}}\right)$ and X_n is given on gens $c_i, \bar{c}_i$

by the div diff op

$$\frac{\xi_1 - \xi_2}{\xi_1 - \xi_2} = 1 \qquad \frac{\xi_2 - \xi_1}{\xi_1 - \xi_2} = -1$$

$$\partial_1(f) := \frac{\overset{1}{f} - s_1 f}{\xi_1 - \xi_2} \quad \circlearrowright \quad \mathbb{Q}[\xi_1, \xi_2] \otimes S_2$$

more generally $\mathcal{E}^a 1_n =$  acts on H_k-pmod by $\otimes$ w

the (H_{k+a}, H_k)-bim

$$H_{k+a,\dots,k+1,k} := H^*(\underbrace{Fl(k, k+1, \dots, k+a, N)}_{a\text{-step flag variety}}, \mathbb{Q})$$

$$= \frac{\mathbb{Q}[c_1 \_ c_k, \xi_1 \_ \xi_a, \bar{c}_1 \_ \bar{c}_{N-k-a}]}{I_{k,k+1 \dots k+a}}$$

& defined analog.

Now the divided diff ops

$$\partial_i := \frac{1 - s_i}{\xi_i - \xi_{i+1}}$$

Fact $\operatorname{Im} \partial_i$ = sym in ξ_i, ξ_{i+1} = $\ker \partial_i$

$\Rightarrow \partial_i^2 = 0$

also deg = -2

For this reason ∂_i are a good choice for X_n

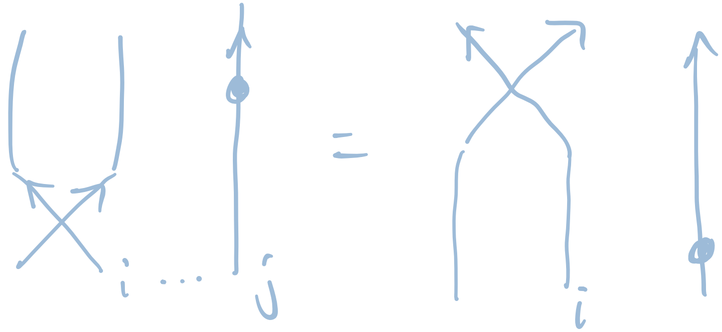
$\langle$ mult by ξ_i, $\partial_i \rangle \subseteq \operatorname{End} \mathbb{Z}[\xi_1 \_ \xi_a]$ is nilHecke

NH_a

Recall NH_a :

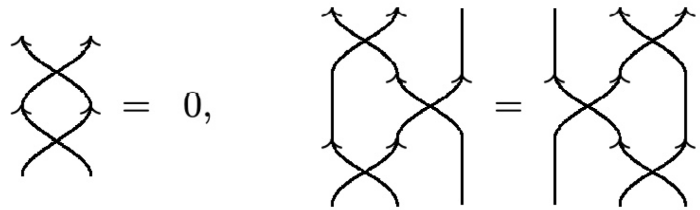
$$\begin{aligned}
&\xi_i\xi_j = \xi_j\xi_i, \\
&\partial_i\xi_j = \xi_j\partial_i \quad \text{if } |i-j|>1, && \partial_i\partial_j = \partial_j\partial_i \quad \text{if } |i-j|>1, \\
&\partial_i^2 = 0, && \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}, \\
&\xi_i\partial_i - \partial_i\xi_{i+1} = 1, && \partial_i\xi_i - \xi_{i+1}\partial_i = 1.
\end{aligned}$$

In terms of diagrams



while the second two lines imply

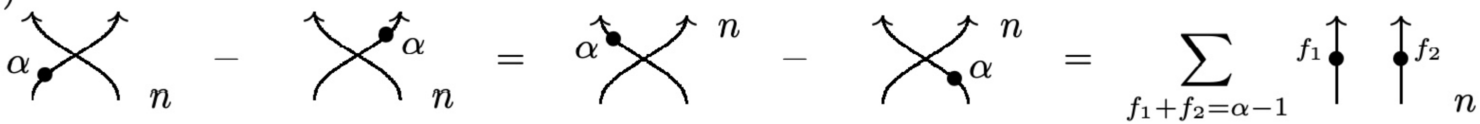
(3.35)



(3.36)

By repeatedly applying (3.36) one can show that the equation

(3.37)



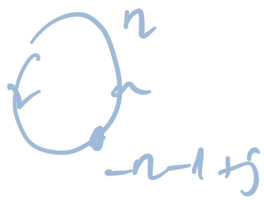
holds.

In this way we see that requiring an action of $\mathcal{U}$ on the cohomology rings of iterated partial flag varieties clarifies the precise form of the relations that should hold on upward oriented strands in $\mathcal{U}$. Using the adjoint structure we get similar relations on downward oriented strands. Bimodule homomorphisms of the appropriate degree can also be found for the cap and cup 2-morphisms in $\mathcal{U}$ [Lau08, Section 7]. These maps turn out to be unique up to a scalar.

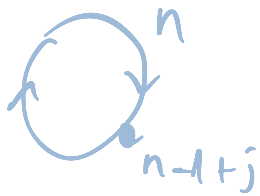
Comput'ns w coho rings of partial flag vars suggest the alg for like-colored strands is governed by NH_a

6. Size of $\mathcal{U}^*$

Prop 8.2



$\mapsto v_{j,n}$ $\qquad n \geq 0$



$\mapsto v_{j,n}$ $\qquad n \leq 0$

$\rightsquigarrow \mathcal{U}^*(1_n, 1_n) \simeq \mathbb{Z}[v_{1,n}, v_{2,n}, \dots]$

gr ring iso w $\deg v_{i,n} = 2i$

$n=0$ case: 2 different isos rel'd by the "∞ Gr rels" (prop 5.5)

Upshot: every closed diagram can be reduced to a unique l.c. of diags of non-nested dotted bubbles w same orient'n.

Let Γ be a closed diagram.

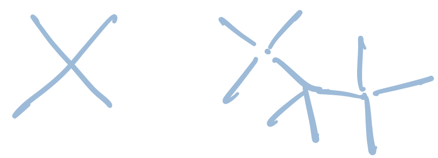
Pf ✓ • use bubble slide eqs in props 5.6 & 5.7 to push dotted bubbles "outside" Γ.

• use gr rels in prop 5.5 to orient outside dotted bubbles the same!

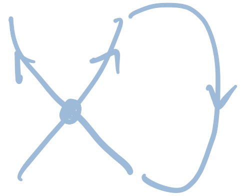
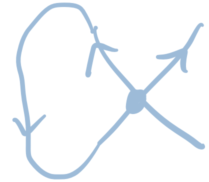
To the left we may have a closed (possibly disconnected) diag (containing no dotted bubbles!)

Take the "innermost" [connected component], and induct on # nilCox gens U_n in it to reduce it to $\sum \circ$:

(contracting ea $||$ to a • and)

By disregarding orientation and dots leaves us w a 4-valent planar graph (3) having at least 1 vertex 

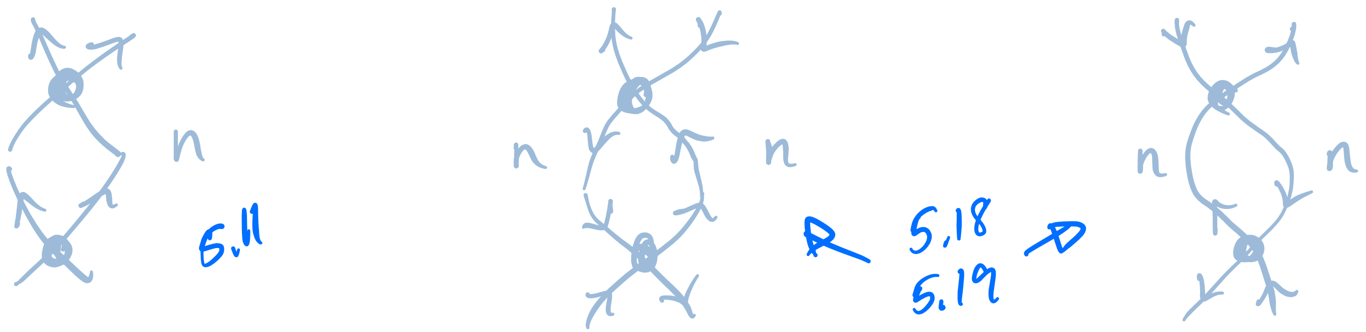
What's a digon face?

• Diagrams of the form  n and n 

(up to dots) produce loops in $\mathcal{G}$.

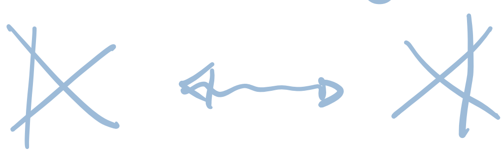
Use "red to wobbles" axioms (5.23 & 5.24)

• Diagrams of the form



produce digon faces in $\mathcal{G}$.

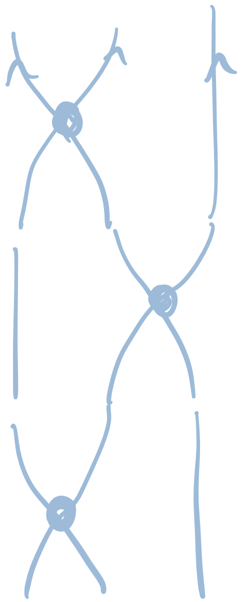
• Graph theory: if a conn planar 4-valent graph has ≥ 1 vertex and no loops or digons then $\exists$ seq of "triangle moves"



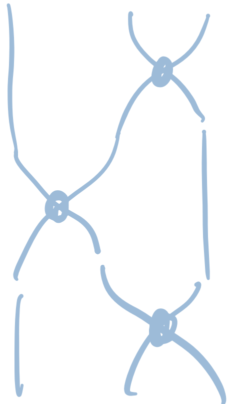
Ex 5.13 & Prop 5.8

transforming it into a diag containing a digon

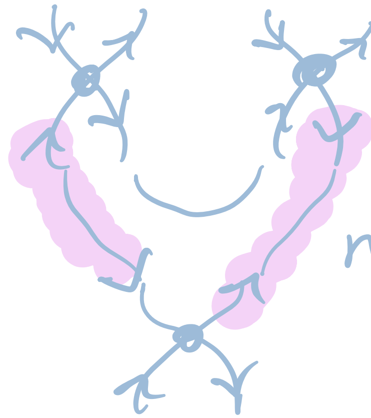
- possibilities come from



n ,

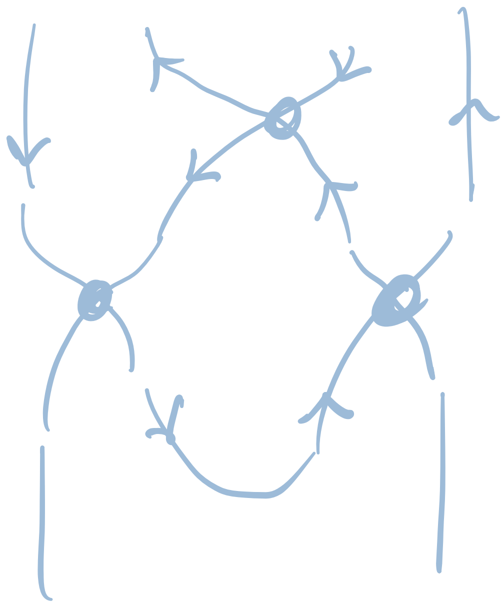


n ,

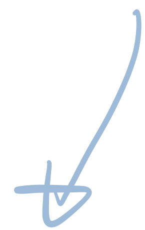


n ?

,



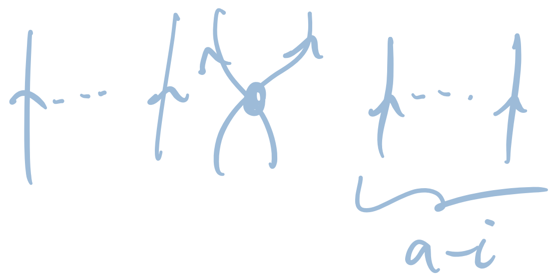
(up to dots)

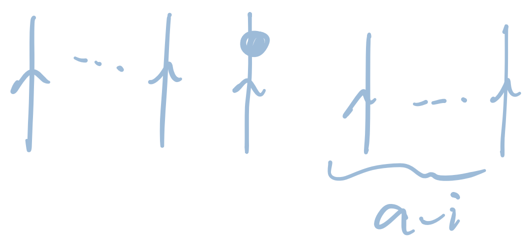


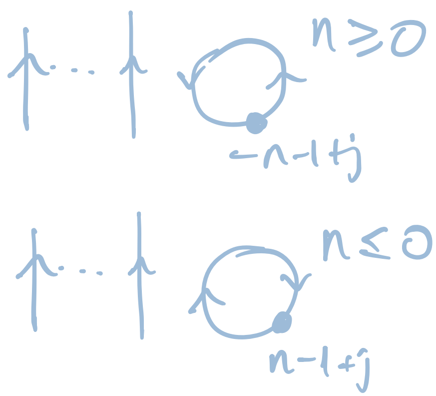
Using nilHecke rels these dots can be moved to the tops of such diags $\rightsquigarrow \sum$ such diags in which # nilCox gens does not incr. ⊠

β

Thm 8.3 $\mathcal{U}^*(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n) \simeq NH_a \otimes \mathbb{Z}[v_{1,n} \dots]$

$n+2a$  $n \longleftarrow u_i \otimes 1$

$n+2a$  $n \longleftarrow x_i \otimes 1$

$n+2a$, $n+2a$ 

$\longleftarrow 1 \otimes v_{i,n}$

in deg $2i$

Pf of thm 8.3

- 8.2 identifies $v_{i,n}$ with dotted bubbles in $\boxed{n}$
- nilHecke action built into def of $\mathcal{U}^*$ $\Rightarrow$ β is a hom
- its image spans $\mathcal{U}^*(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$ $\Rightarrow$ β is surj

Let $y \in \mathcal{U}^*(\mathcal{E}^a 1_n, \mathcal{E}^a 1_n)$. y is a l.c. of diagrams. Let D be a diagram in y let $\mathcal{G}$ be the 4-valent graph got from D by [contracting ea double edge to a pt and] disregarding dots. as in 8.2 $\mathcal{D} = \sum' \mathcal{D}_i$ s.t. $\mathcal{D}_i$ contain no loops or digons, and all nested closed subdiagrams have been reduced to dotted closed bubbles w same orient'n off to one side

Then corresp diagrams D_i can be written as $\sum D_{ij}$ whose graphs $\mathcal{G}_{ij}$ are s.t. no "walk" crosses a given strand more than once

Fact: closed 4-valent connected graphs having ≥ 1 vertex and no loops or digons can be transformed by Δ moves to a diagram containing a digon face.

Injectivity: $f_w(x) \in \mathbb{Z}[x_1, x_2, \dots, x_a]$, $g_w(v) \in \mathbb{Z}[v_1, v_2, \dots]$, $w \in S_a$

sps $\beta\left(\sum f_w(x)\, u_w \otimes g_w(v)\right) = 0$

$N >> 0$: $\Gamma_N($  $) = \sum_{S_n} f_w(\xi)\, \partial_w \otimes G_w = 0$

$f_w(x_i)\,\partial_w : H_{k \dots k+a} \longrightarrow H_{k \dots k+a}$

x_i replaced by ξ_i

$G_w : H_k \to H_k$

$\neq 0$ since come from non-nested dotted bubbles

Claim: $\sum_{S_n} f_w \partial_w = 0$.

- $p_1 \otimes p_2 \in H_{k \dots k+a} \otimes H_k \longmapsto \sum f_w \partial_w(p_1) \otimes G_w(p_2) = 0$

- $v_0 \in S_a$ corresp to min deg term above $\Rightarrow$ $\sum f_w \partial_w(\mathfrak{S}_{v_0}) \otimes \mathfrak{S}_{\bar{w}}(1) = 0$.

- $\partial_w \mathfrak{S}_{v_0} = \begin{cases} \mathfrak{S}_{v_0 w^{-1}} & \ell(v_0 w^{-1}) = \ell(v_0) - \ell(w) \\ 0 & \text{else} \end{cases}$

$\Rightarrow$ only nonzero term is $w = v_0$ term

$\rightsquigarrow f_{v_0} \partial_{v_0}(\mathfrak{S}_{v_0}) \otimes \mathfrak{S}_{\bar{v}_0}(1)$

- $\mathfrak{S}_{\bar{w}} \neq 0 \Rightarrow f_{v_0} = 0$

- inducting on deg conclude $f_w = 0$ all w $\square$

7. Categorification theorem

L11 §3.11

"divided powers signed seq"

let $(\underline{\varepsilon}) = (\varepsilon_1^{a_1} \dots \varepsilon_k^{a_k})$ $\varepsilon_i \in \{+,-\}$ let $\mathcal{E}_{\underline{\varepsilon}} \mathbb{1}_n := \mathcal{E}_{\varepsilon_1}^{(a_1)} \dots \mathcal{E}_{\varepsilon_k}^{(a_k)} \mathbb{1}_n$

Spanning sets

WTS $\gamma : \dot{U} \to K_0(\dot{\mathcal{U}})$ is isomorphism of $\mathbb{Z}[q^{\pm}]$-algs

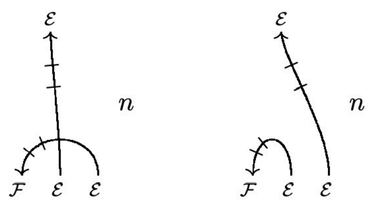
$E_{\underline{\varepsilon}} 1_n \mapsto [\mathcal{E}_{\underline{\varepsilon}} \mathbb{1}_n]$

$\mathcal{E}_{\underline{\varepsilon}} \mathbb{1}_n \Rightarrow \mathcal{E}_{\underline{\varepsilon}'} \mathbb{1}_n$

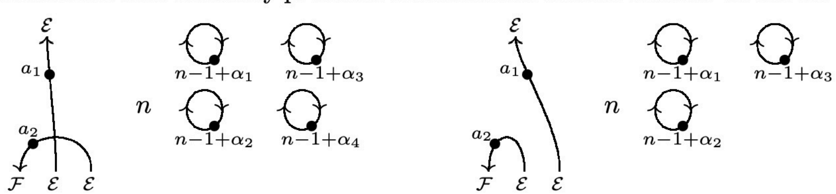
Using rels:

- ✓ no strand intersects itself
- ✓ closed diagrams reduce to non-nested bubbles gathered to the right and having the same orientation
- ✓ dots are confined to a small interval on each strand

Example 3.19. To find the spanning sets for $\mathrm{HOM}_{\mathcal{U}}(\mathcal{F}\mathcal{E}^2\mathbf{1}_n, \mathcal{E}\mathbf{1}_n)$ we chose a small interval on each strand of each possible diagram with no self intersections and no double intersections



The spanning sets are given by these diagrams together with arbitrary number of dots on these intervals and arbitrary products of nonnested dotted bubbles on the far right region.



Showing that these spanning sets are in fact a basis is necessary to prove that our graded 2Homs lifts the semilinear form on $\dot{\mathbf{U}}$.

Need also • Homs not trivial (rels not too strong) $\leftarrow \mathcal{U} \curvearrowright V^N$

• isos lifting $\mathfrak{sl}_2$-rels • 1-mors lifting 1-mors

• indecomposable 1-mor $\longleftrightarrow$ $\dot{B}$ $\leftarrow$ use semilin form

---

"Pf": indec 1-mors of $\dot{\mathcal{U}}$ up to $\{\ \}$ give rise to a basis of $K_0(\dot{\mathcal{U}})$
with str consts in $\mathbb{N}[q^{\pm}]$

• $\operatorname{grdim} \operatorname{Hom}_{\mathcal{U}}(\mathbb{1}_n, \mathbb{1}_n)$ is gen'd by prods of nonnested dotted, same orient'n bubbles

$$= \prod_{a=1}^{\infty} \frac{1}{1-q^{2a}} =: \pi$$

• $\operatorname{grdim} \operatorname{Hom}_{\mathcal{U}}(\mathcal{E}_{\underline{\varepsilon}} \mathbb{1}_n, \mathcal{E}_{\underline{\varepsilon}'} \mathbb{1}_n) = \langle E_{\underline{\varepsilon}} 1_n, E_{\underline{\varepsilon}'} 1_n \rangle \pi$

8. Representation thm   L10 Thm 9.15

The rep $\Gamma_N : \mathcal{U} \to \mathrm{Flag}_N$ yields a rep

$$\dot{\Gamma}_N : \dot{\mathcal{U}} \to \mathrm{Flag}_N$$ and this latter rep

categorifies the irrep V_N of $\dot{U}$

Pf 1) $\underline{\mathrm{Bim}}$ is idempotent complete

2) Idempotents split in Flag_N so

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[dr, "\Gamma_N"'] & \dot{\mathcal{U}} \arrow[d, dashed, "\dot{\Gamma}_N"] \\
 & \mathrm{Flag}_N
\end{tikzcd}$$

3) Γ_N extends to $\dot{\mathcal{U}}$

4) objs of $\mathcal{F}lag_N^*$ are H_k are graded local rings

5) every f.g. proj mod is free

6) H_k has (up to iso & gr sh) unique graded indec proj mod

7) $K_0\left(\bigoplus_{j\geq 0}^{N} \underbrace{H_j\text{-pmod}}_{\text{f.g. gr proj } H_j\text{-mods}}\right)$ is free$/\mathbb{Z}[q^{\pm}]$ of rk $N+1$

$$\rotatebox[origin=c]{-90}{$\rightsquigarrow$} \quad \underset{\mathbb{Z}[q^{\pm}]}{\otimes} \mathbb{Q}(q) \cong V_N$$

8) $\Gamma_N(1_n), \Gamma_N(E1_n), \Gamma_N(F1_n)$

0. Roadmap (Lauda)

Hints that quantum gps are "shadows" of richer str:

- Lusztig's canon basis $\dot{\mathbb{B}}$ of $\dot{U}$
- geo constrs of cat'l qu gp actions
- semilin forms on qu gps having nice props

$\underline{i} = i_1 i_2 \dots i_m \qquad \mathcal{E}_{\underline{i}} = \mathcal{E}_{i_1} \dots \mathcal{E}_{i_m}$ in $\mathcal{C}$

$$R := \bigoplus_{\underline{i},\underline{j}} \mathrm{Hom}(\mathcal{E}_{\underline{i}}, \mathcal{E}_{\underline{j}})$$

Claim $\mathrm{Kar}(\mathcal{C}) \longrightarrow R\text{-pmod}$

$$\mathcal{E}_{\underline{i}} \mapsto P_{\underline{i}} := \bigoplus_{\underline{j}} \mathrm{Hom}(\mathcal{E}_{\underline{i}}, \mathcal{E}_{\underline{j}})$$

is an equivalence of cats

Recall from KL $\gamma : U_{\mathbb{Z}}^{+} \longrightarrow K_0(R)$

$$E_{\underline{i}}^{a} \mapsto [\mathcal{E}_{\underline{i}}^{a}]$$

mult $\mapsto$ Ind

comu $\mapsto$ Res

bilin form $\mapsto$ Hom form

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "2\text{-rep}"] \arrow[d, rightsquigarrow, "K_0"] & \mathcal{V} \arrow[d, rightsquigarrow, "K_0"] \\
\dot{U} \arrow[r, "\text{rep}"] & V
\end{tikzcd}$$