Categorification of tensor products!

1) quantum invariants of knots.
2) categorify all the things!

still somewhat open: general tensor prod of cat. actions. (Rouquier?)

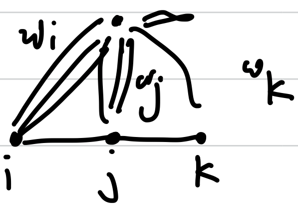
Step 1: categorification of simples.

Consider favorite Cartan matrix A and dominant integral weight λ. Characterized by $\alpha_i^\vee(\lambda) = w_i$.

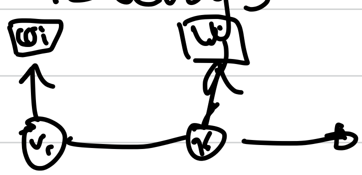
Gives $U(\mathfrak{g})$ and irrep $V(\lambda)$.

Extend Cartan by adding a new root α_∞, w/ $\alpha_i^\vee(\alpha_\infty) = -w_i$. Bigger Lie algebra is $\mathfrak{g}_+$.
$\alpha_\infty^\vee(\alpha_i)$

In terms of Dynkin diagram,



Crawley-Boevey's trick: Nakajima quiver
v_i, w_i



In , F_∞ generates a copy of $V(\lambda)$ under action of F_i's in the adjoint action of $U(\mathfrak{g})$.

$\mathfrak{n}_+$ - lower triangular in $\mathfrak{g}_+$

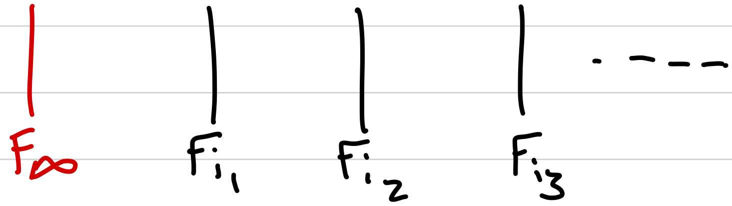
$\mathfrak{n}$ - lower triangular in $\mathfrak{g}$.

$$\frac{U(\mathfrak{n}) F_\infty U(\mathfrak{n})}{U(\mathfrak{n})\mathfrak{n} F_\infty U(\mathfrak{n})} \ni F_\infty \leftarrow \text{highest weight v.}$$

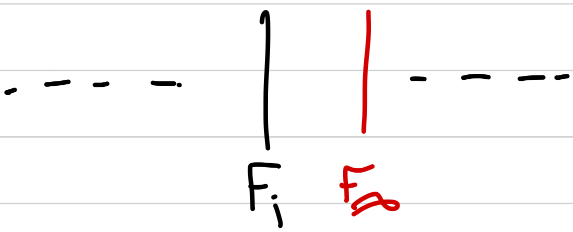
right of weight λ for $U(\mathfrak{g})$ on

$$F_\infty U(\mathfrak{g}) = V(\lambda)$$

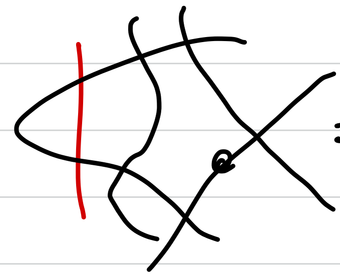
How to categorify? KLR for $\mathfrak{g}_+$, w/ one strand for F_∞



Relation of $/\mathcal{U}(n)$ in F_∞ $\Rightarrow$

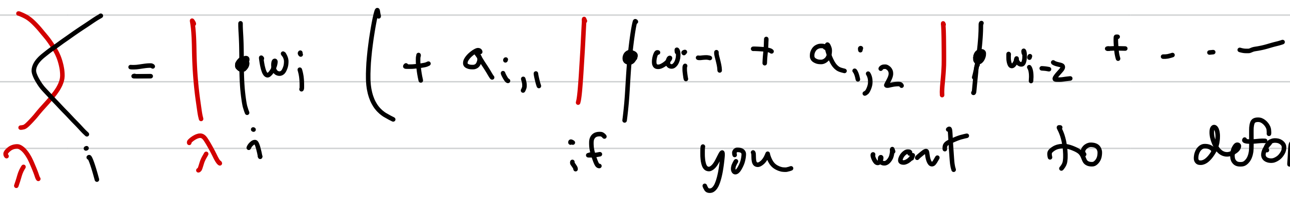


$= 0$. + 2-sided ideal



$= 0$

So, resulting algebra has only KLR diagrams w/ F_∞ on left, and relation



(if you want to deform) $= 0$

Here I'm setting  $= 0$, can absorb into a_i's above.

Cyclotomic quotient!

Generalize to tensor products

Have dominant weights $\lambda_1, \ldots, \lambda_\ell$, can define $\mathfrak{g}_+$ to have ℓ new roots $\infty_1, \ldots, \infty_\ell$

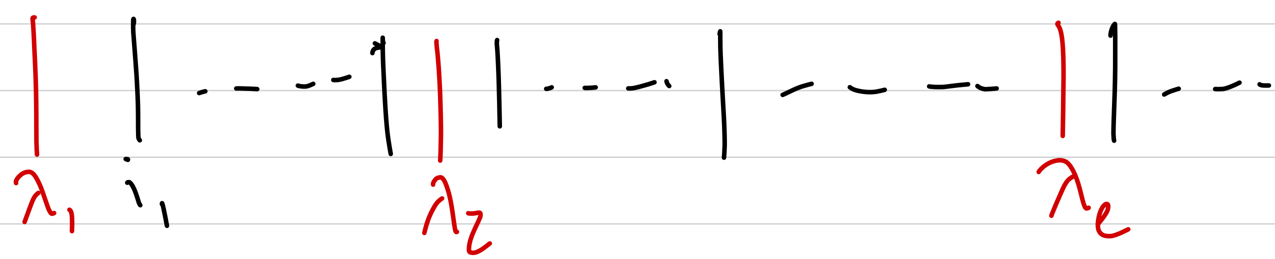
$$\alpha_i^\vee(\alpha_{\infty_k}) = \alpha_i^\vee(\lambda_k) = \lambda_k^i$$

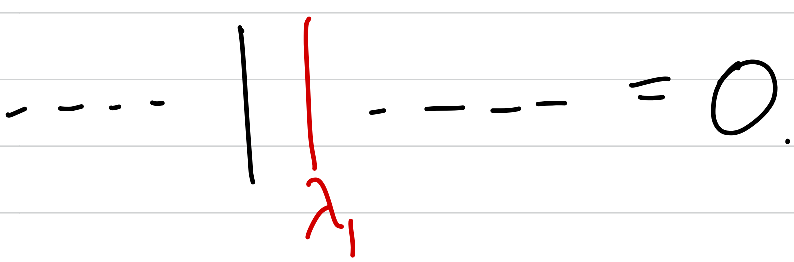
$$V(\lambda_1) \otimes V(\lambda_2) \otimes \cdots \otimes V(\lambda_\ell) \cong \frac{U(n) F_{\infty_1} U(n) F_{\infty_2} U(n) \cdots U(n) F_{\infty_\ell} U(n)}{U(n)\, n\, F_{\infty_1} \cdots F_{\infty_\ell} U(n)}$$

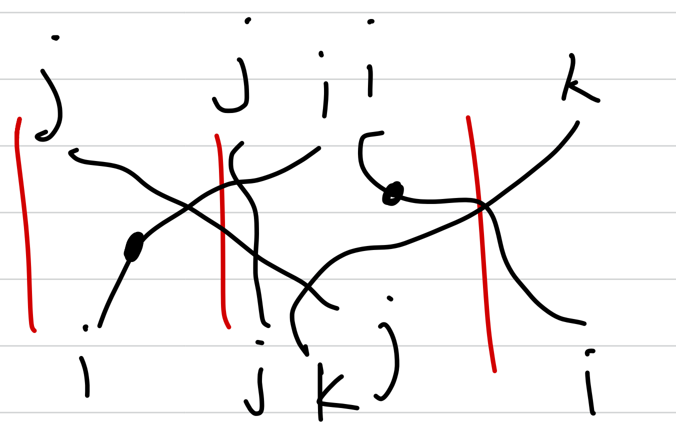
via $F_{\infty_1} u_1 F_{\infty_2} u_2 \cdots F_{\infty_\ell} u_\ell \longmapsto ((v_1 u_1 \otimes v_2) u_2 \otimes v_3) u_3 \otimes \cdots \otimes v_\ell) u_\ell$

$$[F_{\infty_1} u_1 \cdot F_{\infty_2} u_2, x]$$
$$F_{\infty_1} u_1 [F_{\infty_2} u_2, x]$$
$$[F_{\infty_1} u_1, x] F_{\infty_2} u_2$$

Categorification is almost the same.

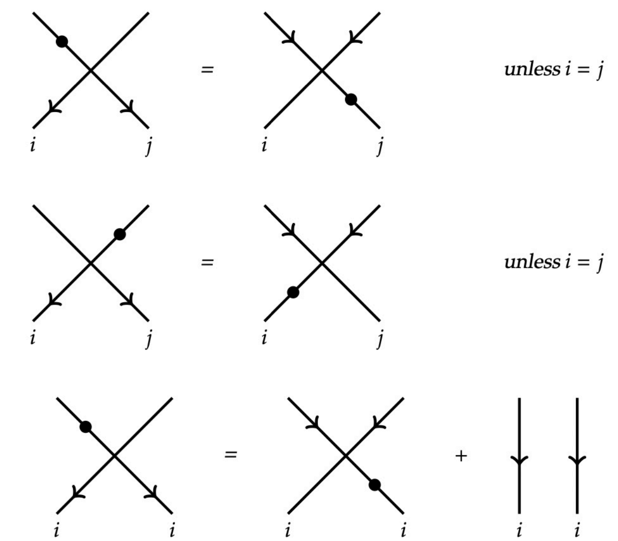




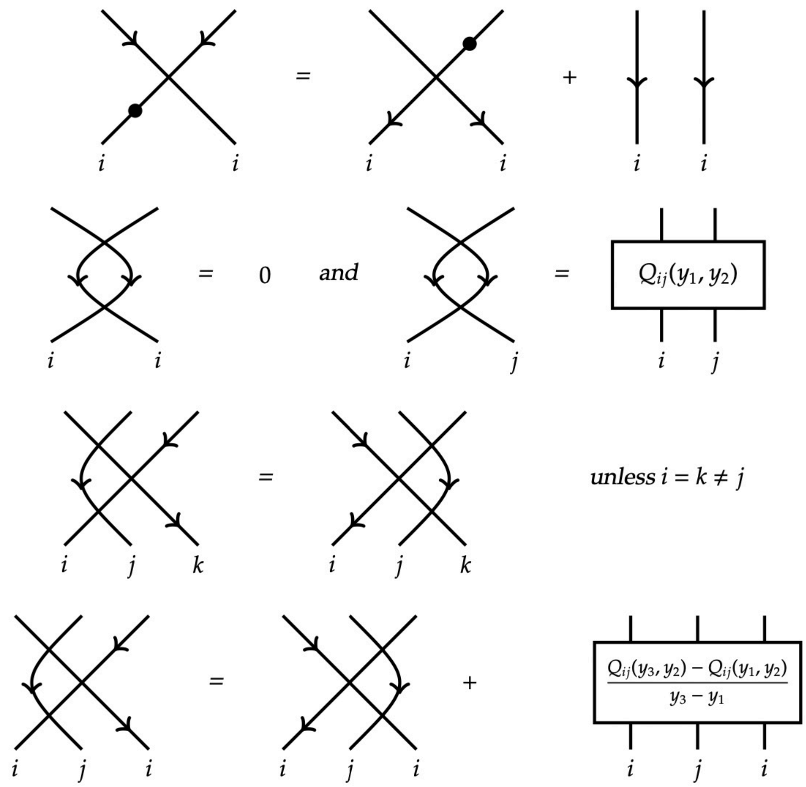


Call this algebra $T^{\underline{\lambda}}$.

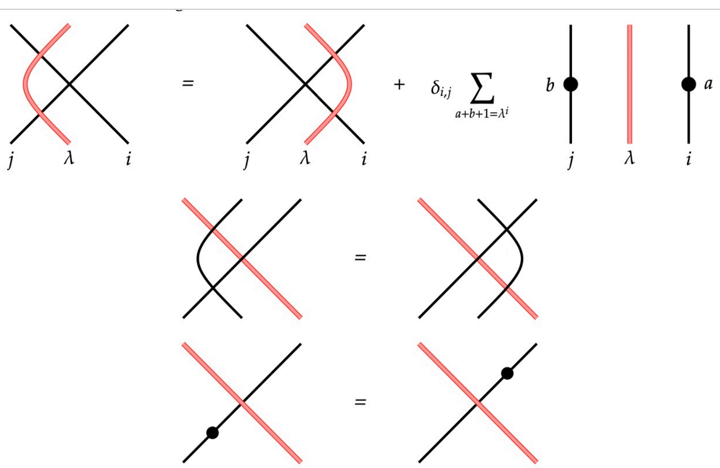
Relations:



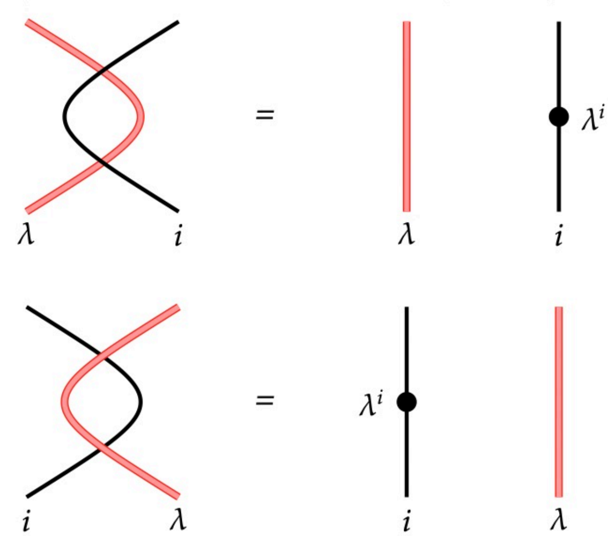
unless i = j
unless i = j



0
and
$Q_{ij}(y_1, y_2)$
unless $i = k \neq j$
$\frac{Q_{ij}(y_3, y_2) - Q_{ij}(y_1, y_2)}{y_3 - y_1}$



$\delta_{i,j} \sum_{a+b+1=\lambda^i}$



λ^i
λ^i

What do we want to know about this algebra?

<u>Thm</u> The category of T^{λ}-modules has a categorical $\mathfrak{g}$-action given by induction and restriction functors.

<u>Pf</u> Not obvious, even for $\ell=1$. Ruined my summer in 2009.
Basic idea: prove "one $\mathfrak{sl}_2$ at a time." If you only Add cyclo rel on strands w/ label i, then E_i, F_i for i define cat action of $\mathfrak{sl}_2$.

T^{λ}_{ν}

$T^{\lambda}_{\nu+\alpha_i}$

$$T^{\lambda}_{\nu} = \sum \lambda_i - \nu = \sum (\#\ i\text{-strands})\,\alpha_i$$

$= 0$

projectives Lusztig integral form

<u>Thm</u> $K^0(T^{\underline{\lambda}}) \cong V(\lambda_1)\otimes \cdots \otimes V(\lambda_\ell)$
$K^0_q(T^{\underline{\lambda}}) \cong V_q(\lambda_1)\otimes \cdots \otimes V_q(\lambda_\ell)$

Not hard once you know about cat action.

$\ell = 1$: $K^0(T^{\underline{\lambda}})$ is gen by a single highest weight vec. of wt. λ. Must be $V(\lambda)$.

$\ell \geq 2$: Show by induction that

$$\mathrm{Hom}(P_{\underline{i}}, P_{\underline{j}}) = \text{inner product on } V(\lambda_1)\otimes \cdots \otimes V(\lambda_\ell)$$

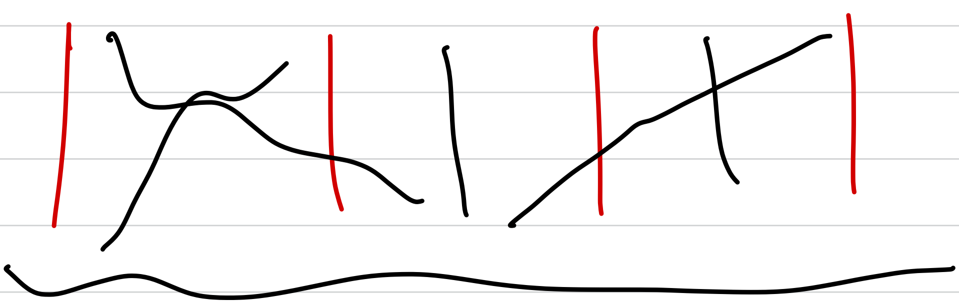
Projectives are obtained by just multiplying by an idempotent on one side or the other, say bottom.

anything

fixed

Thus $\mathrm{Hom}(P_{\underline{i}}, P_{\underline{j}}) =$

$\underline{i}$

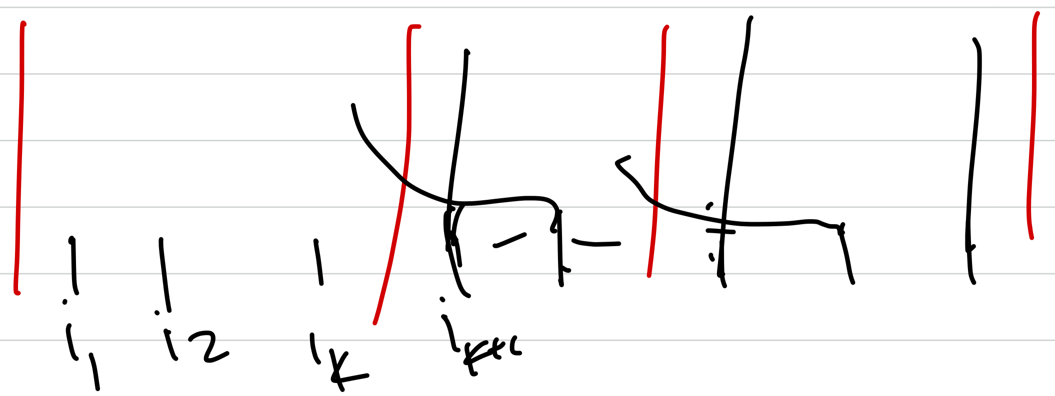


$\underline{j}$

More intuitive proof goes by finding objects corresponding to pure tensors. $v_1 u_1 \otimes v_2 u_2 \otimes \cdots$

These are <u>standard</u> <u>modules</u>. $S_{\underline{i}}$

Standard modules are quotients where we kill any diagram w/ X before any X.



$= 0$

Standard modules are not projective, every projective has a standard filtration!

$F_i, E_i (S_{\underline{i}})$ get a standard filtration.

$S_{\underline{i}}$ $\quad \lambda_1, \nu_1 \quad \lambda_1 - \nu_1 = \sum(\#\text{ i-strands between red 1 and red 2})\alpha_i$

$\quad \lambda_2, \nu_2 \quad \lambda_2 - \nu_2 = \sum(\#\text{ — — red 2 and red 3})\alpha_i$

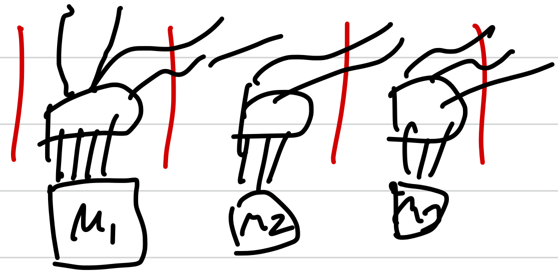
$\vdots \qquad \vdots$

$$R^{\lambda_1}_{\nu_1}, \quad R^{\lambda_2}_{\nu_2}, \quad \ldots \quad R^{\lambda_\ell}_{\nu_\ell}$$

$$\circlearrowleft \qquad \circlearrowleft \qquad \circlearrowleft$$

$$M_1 \qquad M_2 \qquad M_\ell$$

$S(M_1, \ldots, M_\ell) =$



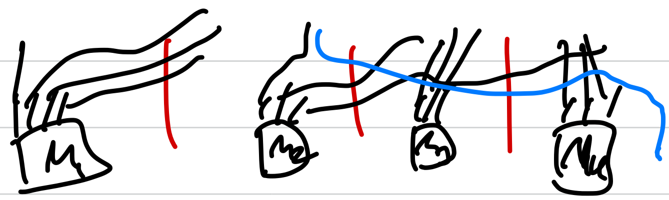
Standard modules $S(P_1, \ldots, P_n)$

$$T^{\underline{\lambda}} \circlearrowright \bigoplus_{\underline{\nu}} S_{\underline{i}} \circlearrowleft R^{\lambda_1}_{\nu_1} \otimes \cdots \otimes R^{\lambda_\ell}_{\nu_\ell}$$

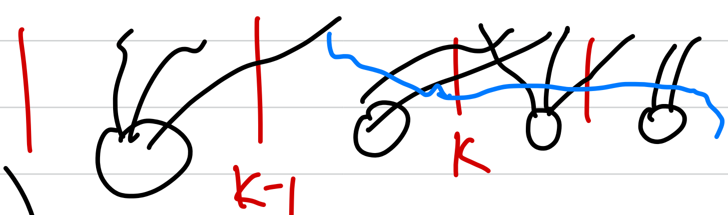
$$P_{i_1} \qquad P_{i_2}$$

$$S(P_{i_1}, \ldots, P_{i_\ell}) = S_{\underline{i}}$$

$F_i S(M_1, \ldots, M_\ell) =$



Subquotients =

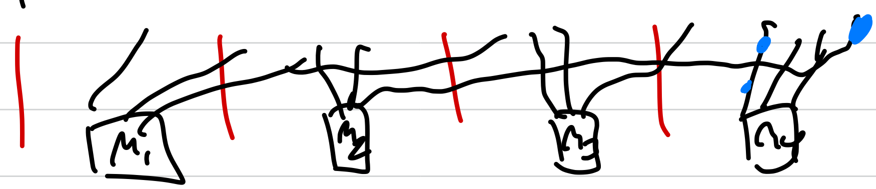


$$S(P_{i_1}, \ldots F_i P_{i_k}, \ldots P_{i_\ell})$$

$[S(M_1, \_\_, M_\ell)] = [M_1] \otimes \cdots \otimes [M_\ell]$

$F_i[S(M_1, \dots, M_\ell)] = \sum q^{?}(M_1) \otimes \cdots \otimes F_i(M_k) \otimes \cdots \otimes [M_\ell]$

$E_i S(M_1, \dots, M_\ell)$



$S(M_1, \dots, E_i M_k, \dots, M_\ell)$

Natural Q: How unique? (w/ Ivan Losev)

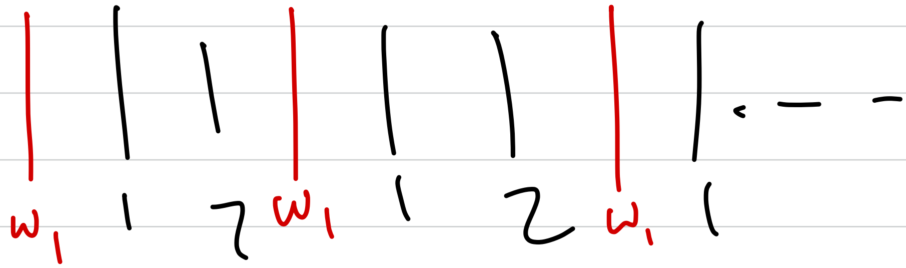
"Tensor product categorification"

1) categorial $\mathfrak{g}$-action
2) notion of standard module (in particular, all projectives are standard filtered)
3) E_i, F_i standards has expected filtration.

Thm Every $\lambda_1, \_\_, \lambda_\ell$ has a unique tensor product cat. (up to equivalence): T^{λ}-mod

[Brundan - Losev - W. $\Rightarrow$ super KL conjecture mult. of simples in Vermas for $\mathfrak{gl}(m|n)$, $(\mathbb{C}^\infty)^{\otimes m} \otimes ((\mathbb{C}^\infty)^*)^{\otimes n}$

Cat $\mathcal{O}$ for $\mathfrak{gl}_n$, regular block



n times

$n-1$ 1's

$n-2$ 2's

$n-3$ 3's

$V(-\rho)$