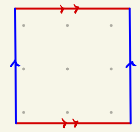
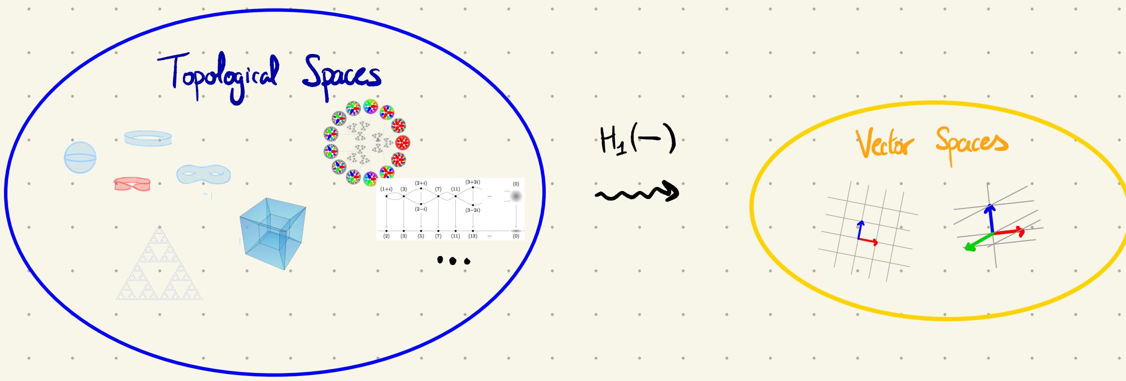
6. Cellular homology: what is a hole?

Recall that  $\not\cong$ , because the torus has a hole, but what does this mean precisely?

For instance, where is the hole in  ?

We answer this today, and by the end you will be convinced that the torus has 2 holes!

Idea: Space X $\rightsquigarrow$ "Vector space" $H_1(X)$

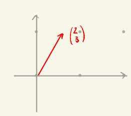


But... what is a vector space?

Lightning introduction to linear algebra

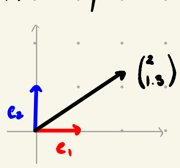
A vector space V with a basis $b_1, \dots, b_n$ is like $\mathbb{R}^2$ with its basis $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$:

0) V is a set, whose elements are called vectors: $\begin{pmatrix}2\\3\end{pmatrix}$ is a vector in $\mathbb{R}^2$.



1) In $\mathbb{R}^2$ you can add vectors: $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$, and scale them: $3 \cdot \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix}$.

2) In $\mathbb{R}^2$ every vector can be written as $\lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix}$, for some numbers λ_1, λ_2



$\begin{pmatrix}2\\1.3\end{pmatrix} = 2 \begin{pmatrix}1\\0\end{pmatrix} + 1.3 \begin{pmatrix}0\\1\end{pmatrix}$

We say "$\begin{pmatrix}2\\1.3\end{pmatrix}$ is a linear combination of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$."

3) Step 2 can only be done in a unique way: $\begin{pmatrix}2\\1.3\end{pmatrix} \longleftrightarrow 2, 1.3$

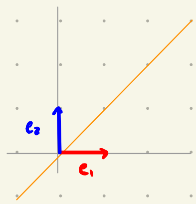
Examples:
- $\mathbb{R}^2$ with basis $b_1 = \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\3\end{pmatrix}$
- Let $S = \{$oatmilk, cereal$\}$. Then $V = \{$linear combinations of oatmilk and cereal$\}$ $= \mathrm{Span}(S)$ is a vector space with basis $b_1 =$ oatmilk, $b_2 =$ cereal
- Let $S = \{$red, green, blue$\}$. Then $V = \mathrm{Span}(S)$ has basis red, green, blue.

Vector subspaces

A subspace of V is a subset $W \subseteq V$ such that if $w_1, w_2 \in W \Rightarrow \lambda_1 w_1 + \lambda_2 w_2 \in W$ (for numbers $\lambda, \lambda_1, \lambda_2$)

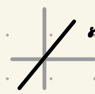
$w \in W \Rightarrow \lambda w \in W$

Example: the line $y = x$ in $\mathbb{R}^2$



$W = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$

Basis: $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (There are more choices!)

Fact: every subspace of $\mathbb{R}^2$ is either $\begin{cases} \mathbb{R}^2 \\ \text{a line} \\ \text{the origin} \end{cases}$ → example: 

. Basis?

Example: $V = \mathbb{R}^3$, $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } z = x + y \right\}$. Question: what is a basis of W?

↳ View GeoGebra

Definition: the number of vectors of any basis of a vector space is called its dimension. We write $\dim(V)$.

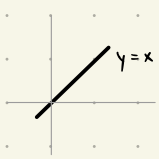
Vector quotients (Subtle!!)

Given a vector space V and a vector subspace $W \subseteq V$, one can form the quotient vector space V/W as follows:

Consider the equivalence relation $\sim$ on V given by: $v_1 \sim v_2 \Leftrightarrow v_1 - v_2 \in W$. Then $V/W = V/\sim$

Fact: V/W is a vector space! Let's see some examples. (Quick review of $X/\sim$)

Example:

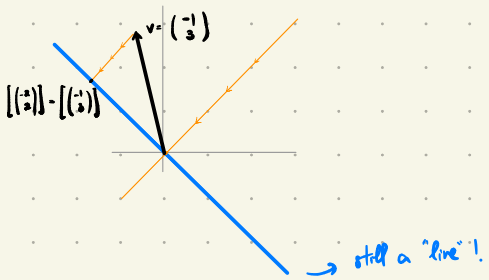
$V = \mathbb{R}^2$, $W =$ 

Equivalence class of $v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$:

$$\begin{aligned}
\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right] &= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in W \right\} \\
&= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} a \\ b-2 \end{pmatrix} \in W \right\} \\
&= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : a = b-2 \right\} \\
&= \left\{ \begin{pmatrix} a \\ a+2 \end{pmatrix} : a \in \mathbb{R} \right\}
\end{aligned}$$

Notice: we can write this as $\begin{pmatrix} 0 \\ 2 \end{pmatrix} + W$

What is happening pictorially? We are "collapsing" the line $y=x$, as well as its parallel lines:



Quotients are useful when we want to get rid of information. For instance, if $V = \text{Span}(\{\text{oatmilk}, \text{cereal}\})$, $W = \text{Span}(\{\text{cereals}\})$

Then V/W is the 1-dimensional space where we "ignore" the cereal: $[2 \cdot \text{oatmilk} + \text{cereal}] = [2 \cdot \text{oatmilk} + 3 \cdot \text{cereal}]$

Linear algebra on Sage

Computing bases for subspaces and quotients is not always easy by hand. Here are some examples of computations on Sage:

- Basis of a subspace:
- Basis of a quotient:
- Basis of a subquotient:

In [13]:
```
V = QQ^3

print('Basis of subspace:')
W0 = V.span([V.0+V.1, V.2])
print(W0.basis())

W1 = V.span([V.0+V.1+V.2])

print('Basis of quotient:')
Q0 = V/W1
for v in Q0.basis():
    print(Q0.lift(v))

print('Basis of subquotient')
Q1 = W0/W1
for v in Q1.basis():
    print(Q1.lift(v))
```

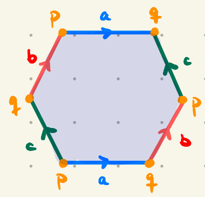
Out[13]:
```
Basis of submodule:
[
(1, 1, 0),
(0, 0, 1)
]
Basis of quotient:
(1, 0, 0)
(0, 1, 0)
Basis of subquotient
(1, 1, 0)
```

In [21]:
```
M = matrix(QQ, [[1,1,1]])
K = A.right_kernel()
K.basis()
```

Out[21]:
```
[
(1, 0, -1),
(0, 1, -1)
]
```

Cellular homology

Recall that we have been thinking of surfaces such as

$X =$ 

We can think of this as three sets: $X^0 = \{ p, q \}$ $\rightsquigarrow$ form $C_0 = \text{Span}(\{p, q\})$

"Cell structure" $X^1 = \{ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \}$ $\rightsquigarrow$ form $C_1 = \text{Span}(\{ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \})$

$X^2 = \{ t \}$ $\rightsquigarrow$ form $C_2 = \text{Span}(\{ w \})$

Define: • the boundary of $\overrightarrow{a}$ is $\partial(\overrightarrow{a}) = q - p$

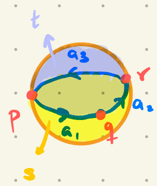
$\overrightarrow{b}$ is $\partial(\overrightarrow{b}) = p - q$ $\Rightarrow$ This gives us a map $\partial_1 : C_1 \to C_0$

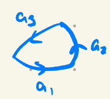
$\overrightarrow{c}$ is $\partial(\overrightarrow{c}) = q - p$ for instance, $\partial_1(a + 2b) = \partial_1(a) + 2\partial_1(b)$
$= p - q$

• the boundary of t is $\partial(t) = -a - b - c + a + b + c = 0$

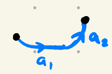
We are ready to define H_1!

A cycle is an element v of V_1 such that $\partial_1(v) = 0$

Example: on 

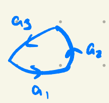
the boundary is $\partial(a_1 + a_2 + a_3) = (q - p) + (r - q) + (p - r) = 0$ $\Rightarrow$ 

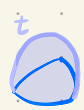
is a cycle

Notice: $\partial(a_1 + a_2) = (q - p) + (r - q) = r - p \neq 0$ $\Rightarrow$ 

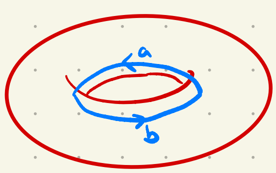
not a cycle

A boundary is an element v of V_1 such that $v = \partial_2(w)$ for some $w \in C_2$.

Example: $a_1 + a_2 + a_3 = \partial(t)$ $\Rightarrow$ 

is a boundary, namely the boundary of 

In fact, every boundary is a cycle: the boundary of a disk is always a circle! But not every cycle is a boundary:

In 

$a + b$ is clearly a cycle, but it is no one's boundary

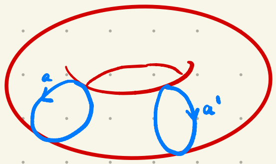
it's a hole!

Question: What is a second, different "hole"?

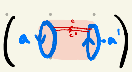
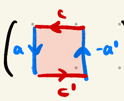
At this point it is tempting to make the following definition:

"The holes of X are the cycles which are not boundaries."

However, there is a problem:



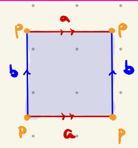
These two cycles represent the same "hole". How to fix this? Notice that although a and a' are not boundaries,

$a - a' = \partial\Big($  $\Big) = \partial\Big($  $\Big)$

In other words, the holes of X are encoded in the vector space

$$\boxed{H_1(X) = \text{Cycles of } X \big/ \text{Boundaries of } X}$$

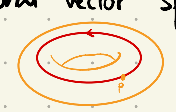
Shorthand: $Z(X) = \{ v \in V_1 : \partial_1(v) = 0 \}$
$B(X) = \{ v \in V_1 : v = \partial_2(w) \text{ for some } w \in V_2 \}$

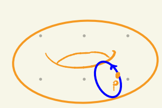
Example: $H_1(T^2)$ $\quad T^2 =$ 

Cell structure: $X^0 = \{ p \}$
$X^1 = \{ a, b \}$
$X^2 = \{ t \}$

$Z(T^2) = \text{Span}(\{ a, b \}) = V_1$ since $\partial(a) = \partial(b) = p - p = 0$

$B(T^2) = \text{Span}(\{0\}) = 0$ since the only boundary is $a + b - a - b$

$\Rightarrow H_1(T^2)$ is $\quad Z(T^2) \big/ B(T^2) = V_2 / 0 = V_2$, a 2-dimensional vector space with basis $a =$ 

$b =$ 

Remark: one defines $H_2, H_3, \ldots$ similarly: H_2 counts "2-dimensional holes". For instance, $H_2(S^2)$ is 1-dimensional.

Theorem: the dimension of H_i is independent of the choice of cellular structure of X.

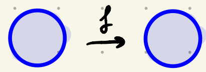
Extra: Brouwer's fixed point theorem

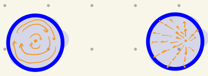
The importance of H_1 is not so much as a numerical invariant ($\dim H_1$) but rather due to the following fact:

For any continuous map $f : X \longrightarrow Y$ there is a linear map $H_1(f) : H_1(X) \to H_1(Y)$

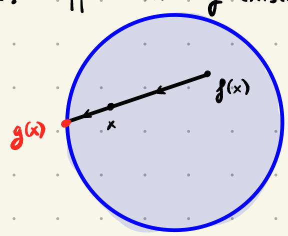
This respects composition: $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad H_1(g \circ f) = H_1(g) \circ H_1(f)$ and $H_1(id_X) = id : H_1(X) \to H_1(X)$

This has the following application:

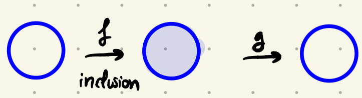
Theorem (Brouwer): any continuous map  has a fixed point: $f(x_0) = x_0$

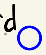
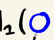
Examples: 

How do you even prove something like this? Suppose such an f existed.

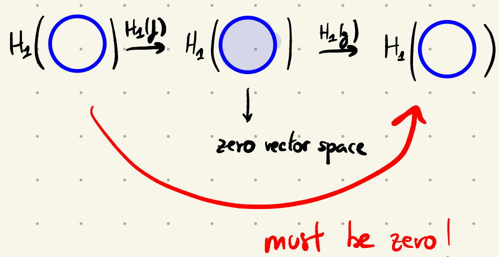
Clever idea: consider the map g: 

Since $f(x) \neq x$ ever, g is well defined.

Now, we have maps 

Observe that $g \circ f = id$, therefore $H_1(g) \circ H_1(f) = H_1(g \circ f) = H_1(id) = id : H_1($$) \to H_1($$)$

dim 1   dim 1

On the other hand we have 

Contradiction!

5. Cellular homology: what is a hole?

1. Verify the following computations on Sage.

For the following, feel free to use Sage

2. Compute ∂_1 and ∂_2 for P^2. Then, compute $H_1(P^2)$

3. Compute $H_1(T^2 \# T^2)$ and $H_1(P^2 \# P^2)$ (Challenge: what is $H_1(X \# Y)$ for surfaces X, Y?)

4. Give cell structures for the cylinder and the Möbius strip, indicating ∂_1 and ∂_2 in each case. Then compute their H_1.

5. Compute some examples of $H_0(X)$ and give an interpretation for it. (Consider e.g. $S^2 \sqcup S^2$)