4. Equivalence classes: how to glue in math

1. The set difference of X and Y is defined as: $X \setminus Y = \{x \in X \mid x \notin Y\}$. Prove the following equalities:

   a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

   b) $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$

   Draw their associated Venn diagrams

2. Decide which of the following relations are equivalence relations. Indicate the equivalence classes when this is the case.

   a) $(\mathbb{R}, \sim)$, where $x \sim y$ means that x and y have the same sign

   b) $(\mathbb{R}, \sim)$, where $x \sim y$ means that $x \geq y$

   c) $(\mathbb{Z}, \sim)$, where $x \sim y$ means that $x - y$ is even.

3. Let $Cyl = \{(x,y,z) \mid x^2+y^2=1, -1 \leq z \leq 1\}$ be the cylinder . Describe $Cyl/\sim$ as best as you can for the following equivalence relations:

   a) $(x,y,1) \sim (x',y',1)$ for all x,y,x',y'

   b) $\begin{cases} (x,y,1) \sim (x',y',1) & \text{for all } x,y,x',y' \\ (x,y,0) \sim (x',y',0) & \text{for all } x,y,x',y' \end{cases}$

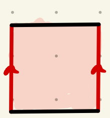
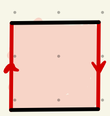
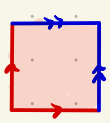
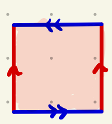
   c) $(x,y,1) \sim (x,y,0)$ for all x,y

   d) $(x,y,z) \sim (x,y,z')$ for all x,y,z,z'

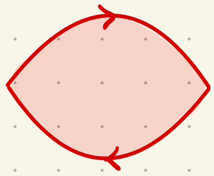
   e) $(x,y,z) \sim (-x,y,z)$ for all x,y,z

   f) $(x,y,z) \sim (-x,-y,z)$ for all x,y,z

4. Identify the following diagrammatically presented quotients of the unit square:

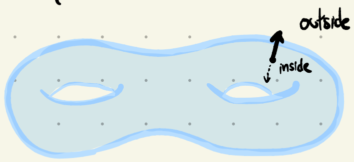
   a)  b)  c)  d) 

5. Wrap your head around the following space:



6. (Challenge) Prove that the space in 5 cannot correspond to a closed surface in $\mathbb{R}^3$.

Hint: a surface in $\mathbb{R}^3$ has, at each point, an "outside" and an "inside" direction, for instance:



Why can't we assign such a direction at each point of the surface of 5 in a "coherent" way?