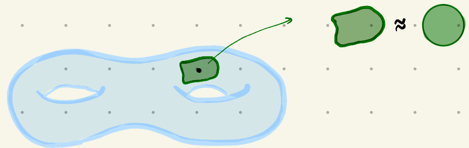
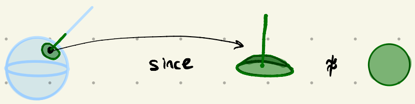
2. Intro to manifolds and knot theory

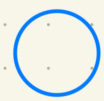
In topology, the main object of study are <u>manifolds</u>. These are spaces such that zooming in on any given point, they look like a line (1-manifold, or curve), a plane (2-manifold, or surface), 3D-space (3-manifold), etc.

Example: A 2-manifold:



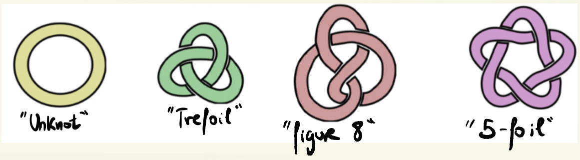


There are two 1-manifolds up to homeomorphism: (Guess)

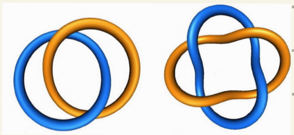
- The infinite real line  "non-closed"
- The circle, or S^1:  "closed"
- Disjoint unions of lines and circles.

Proof that the line and the circle are not homeomorphic: if you remove any point from the line, it breaks up into two components. If you remove any point from the circle, it still has one component.

Since 1-manifolds up to homeomorphism are so easy, let's make them more interesting. Consider closed 1-manifolds embedded into 3D-space. These are called links:



Knots (homeomorphic to the circle)

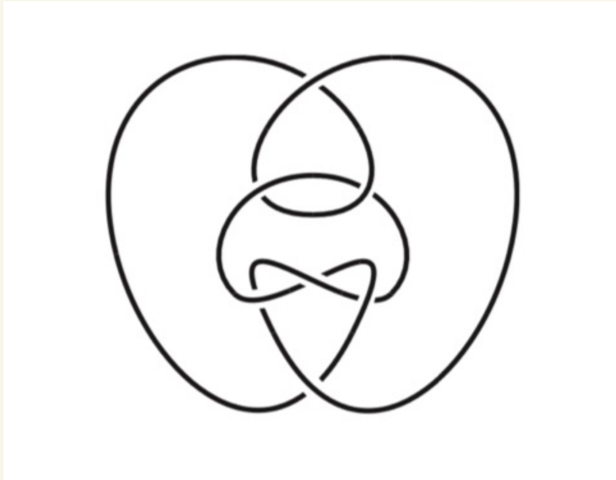


"Hopf link" "Whitehead link"

links (homeomorphic to 1 or more circles)

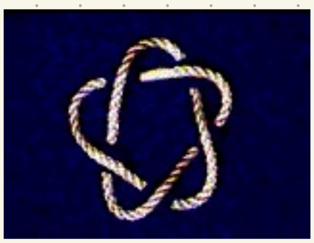
We will consider links up to "isotopy". Roughly, this means that not only the links map to each other, their <u>complements</u> do too.

Poll: is this a link diagram for the unknot?



Poll: which of these can be manipulated into trefoils?

a.



b.

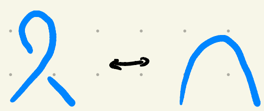
c.

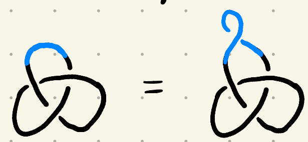
d.

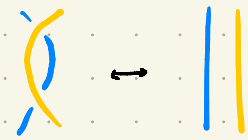
http://sites.oglethorpe.edu/knottheory/wanted/

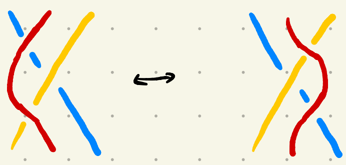
We will typically represent links by planar diagrams like the ones above. It is relatively easy two links are isotopic (if they are):

Theorem (Reidemeister): two link diagrams represent the same Knot if and only if they are related by Reidemeister moves.

RI : 

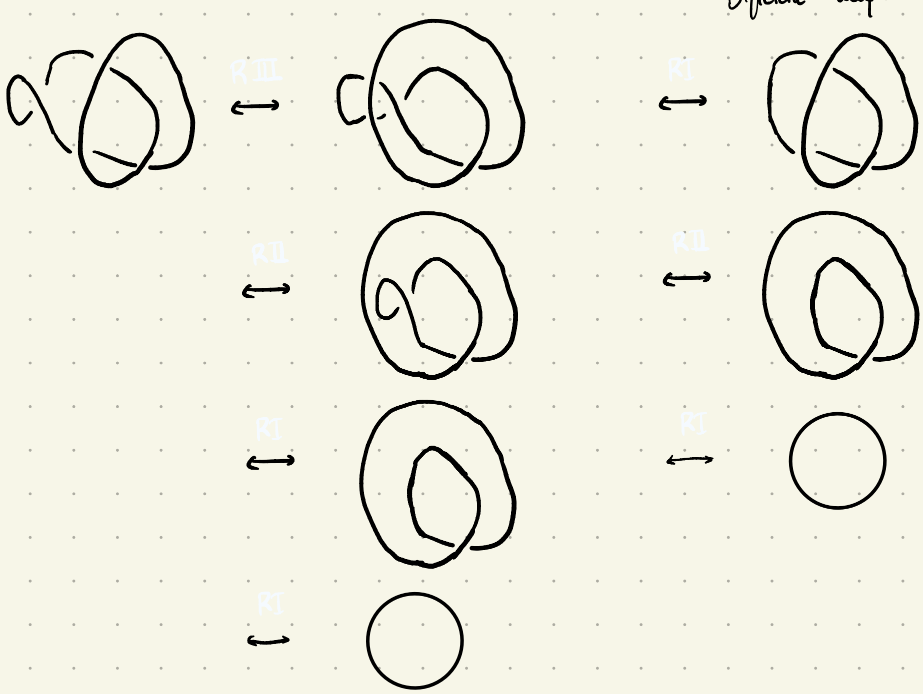
Example: 

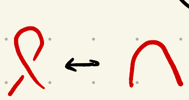
RII : 

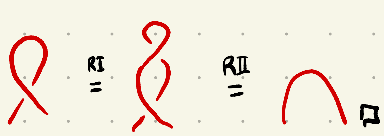
RIII : 

Example:

Different way:



Theorem: RI' :=  follows from RI, RII, RIII.

Proof: 

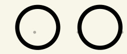
Exercise

2. Intro to manifolds and Knot theory

0. Find:

   a) A link diagram for the unknot with n crossings, for any $n \geq 1$.

   b) A link diagram with n components $(n \geq 2)$ and $2(n-1)$ crossings. (The link must be connected, e.g.  doesn't count.)

   c) A link with three components such that removing any one component yields two separate unknots: 

1. Simplify the following diagrams until obtaining the unknot, indicating each Reidemeister move.

a) 

b) 

c) 

2. You're handed a link diagram for a knot (1 connected component). You know it only has one crossing. Does the knot have to be the unknot? What if it has 2 crossings? What if it has 3?

3. (Optional) Classify links with 2 components which have a diagram with 2 crossings.