Spaces and symmetries (Week 1)

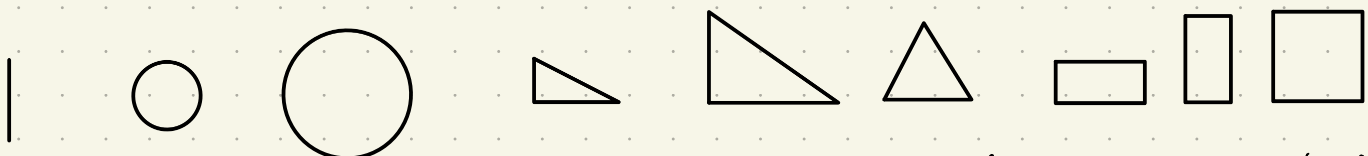
0. Introductions (everyone) + survey

1. Intro to topology

Notions of space in mathematics:

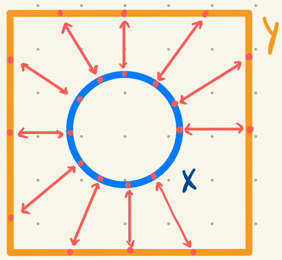
Planar spaces (sets of points in $\mathbb{R}^2$)

Which of the following spaces are "the same"?



(Discuss geometry vs topology)

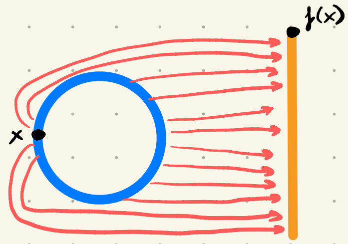
In topology, distances don't matter, two spaces are "homeomorphic" if there is a continuous correspondence between them:



Correspondence: each point in X gets mapped to one point in Y and vice-versa.

Equivalently: $f: X \to Y$ and $g: Y \to X$ are inverse to each other

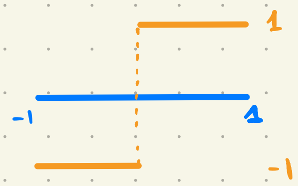
Continuous: • Informally: the map $X \to Y$ doesn't "break" it or "glue" it, and vice-versa



Not continuous correspondence

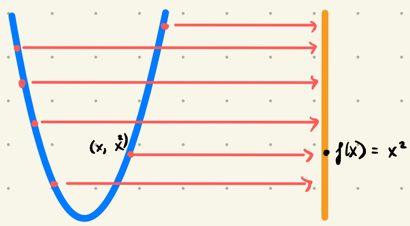
• Formally: for every $y = f(x)$ in Y, and neighborhood around y (= points within a small distance of y) there exists a neighborhood around x (= points within a small distance of x) that gets mapped to the neighborhood around y.

Example: a map which is not continuous: $X = [-1, 1]$, $Y = 1, -1$ $\quad f(x) = \text{sign}(x)$



Example: A continuous mapping which is not a correspondence:

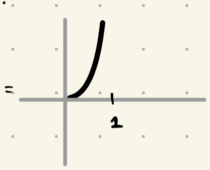
$f((x, x^2)) = x^2$



Example: prove that the line segment $(0,1)$ is homeomorphic to the segment of the parabola $y = x^2$ between 0 and 1.

Step 1: Identify the spaces:

$X =$ 

, $Y =$ 

Step 2: Find a continuous map $f: X \to Y$

My map: $f(x) = (x, x^2)$

(Tip: polynomials, exponentials, square roots, sin, cos, ... are all continuous)

Step 3: Find a continuous map $g: Y \to X$

My map: $g(x, y) = \sqrt{y}$

Step 4: Check that $g \circ f = id_X$

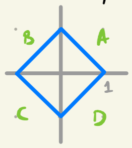
$$f \circ g = id_Y$$

$$g \circ f(x) = g(f(x)) = g(x, x^2) = \sqrt{x^2} \overset{x>0}{=} x \quad ✓$$

$$f \circ g(x, x^2) = f(\sqrt{x^2}) \underset{x>0}{=} f(x) = (x, x^2) \quad ✓$$

Example: Prove that a square and a circle are homeomorphic [P]

Step 1: Identify the spaces:

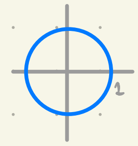
$X =$ 

Four segments: A: $(t, 1-t)$ t from 0 to 1

B: $(t-1, t)$ t from 0 to 1

C: $(t-1, -t)$ t from 0 to 1

D: $(t, t-1)$ t from 0 to 1

$Y =$ 

(x,y) such that $x^2+y^2=1$

$\Leftrightarrow y = \pm\sqrt{1-x^2}$

Step 2: Find a continuous map $f: X \to Y$

$f_A(x,y) = (x, \sqrt{1-x^2})$

$f_B(x,y) = (x, \sqrt{1-x^2})$

$f_C(x,y) = (x, -\sqrt{1-x^2})$

$f_D(x,y) = (x, -\sqrt{1-x^2})$

Continuity at the endpoints: $f_A(0,1) = (0,1) = f_B(0,1)$

$f_B(-1,0) = (-1,0) = f_C(-1,0)$

$f_C(0,-1) = (0,-1) = f_D(0,-1)$

$f_D(1,0) = (1,0) = f_A(1,0)$

Step 3: Find a continuous map $g: Y \to X$

$g_A(x,y) = (x, 1-x)$

$g_B(x,y) = (x, x+1)$

$g_C(x,y) = (x, -1-x)$

$g_D(x,y) = (x, -1+x)$

Continuity at the endpoints: $g_A(0,1) = (0,1) = g_B(0,1)$

$g_B(-1,0) = (-1,0) = g_C(-1,0)$

$g_C(0,-1) = (0,-1) = g_D(0,-1)$

$g_D(1,0) = (1,0) = g_A(1,0)$

Step 4: Check that $g \circ f = id_X$

$f_A \circ g_A(x, \sqrt{1-x^2}) = f_A(x, 1-x) = (x, \sqrt{1-x^2})$ ✓

Exercises

Exercises

1. Intro to topology

1. Show that the following 1-manifolds are homeomorphic.

   a) The unit interval $(0,1)$

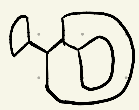
   b) The interval $(3,5)$

   c) (Challenge) The interval (a,b), for $a<b$

2. Argue that if two spaces X and Y are homeomorphic, and in turn another two spaces Y and Z are homeomorphic, then X and Z are homeomorphic.

3. Prove that the (empty) triangle X with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and the (empty) triangle Y with vertices $(0,0)$, $(1,0)$ and $(2,1)$ are homeomorphic. Challenge: prove this for their interiors too.

4. Prove that the "tripod"  is not homeomorphic to the unit interval. (Hint: what happens when we remove the origin)

5. (Challenge) Recall that 1-manifolds are spaces that "locally look like an interval". Define a "web" to be a space that locally looks like either an interval, or a tripod, for instance: . Find two webs W_1, W_2 with two "tripod points" each such that W_1 and W_2 are not homeomorphic. How do you know they aren't?