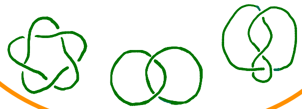
Reminder of last time:

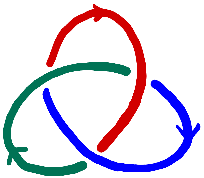
Links

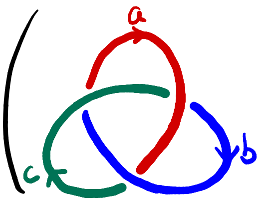


$\rightsquigarrow$

Groups

C_n S_n $\mathbb{Z}$ $\mathrm{Sym}(\square)$

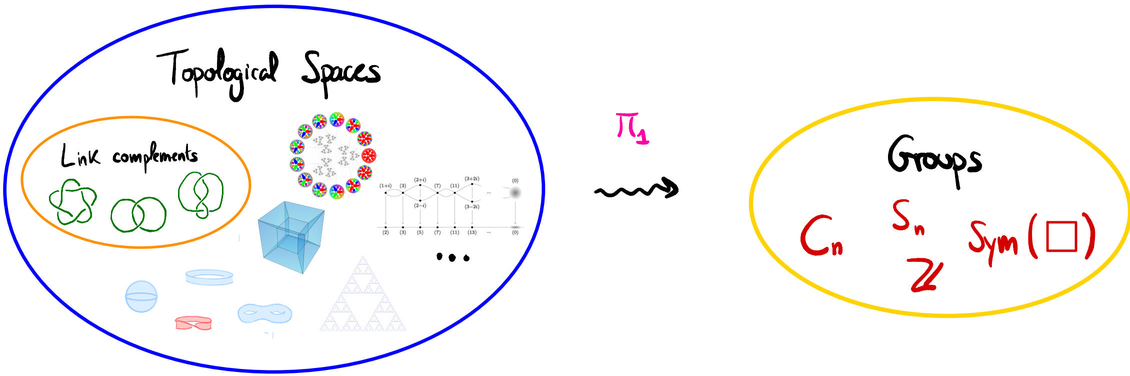


$\pi\Big($ 

$\Big) =$

$$\langle a, b, c \mid a = c^{-1}bc,\ b = a^{-1}ca,\ c = b^{-1}ab \rangle$$

There is a more general notion of π, which extends to:



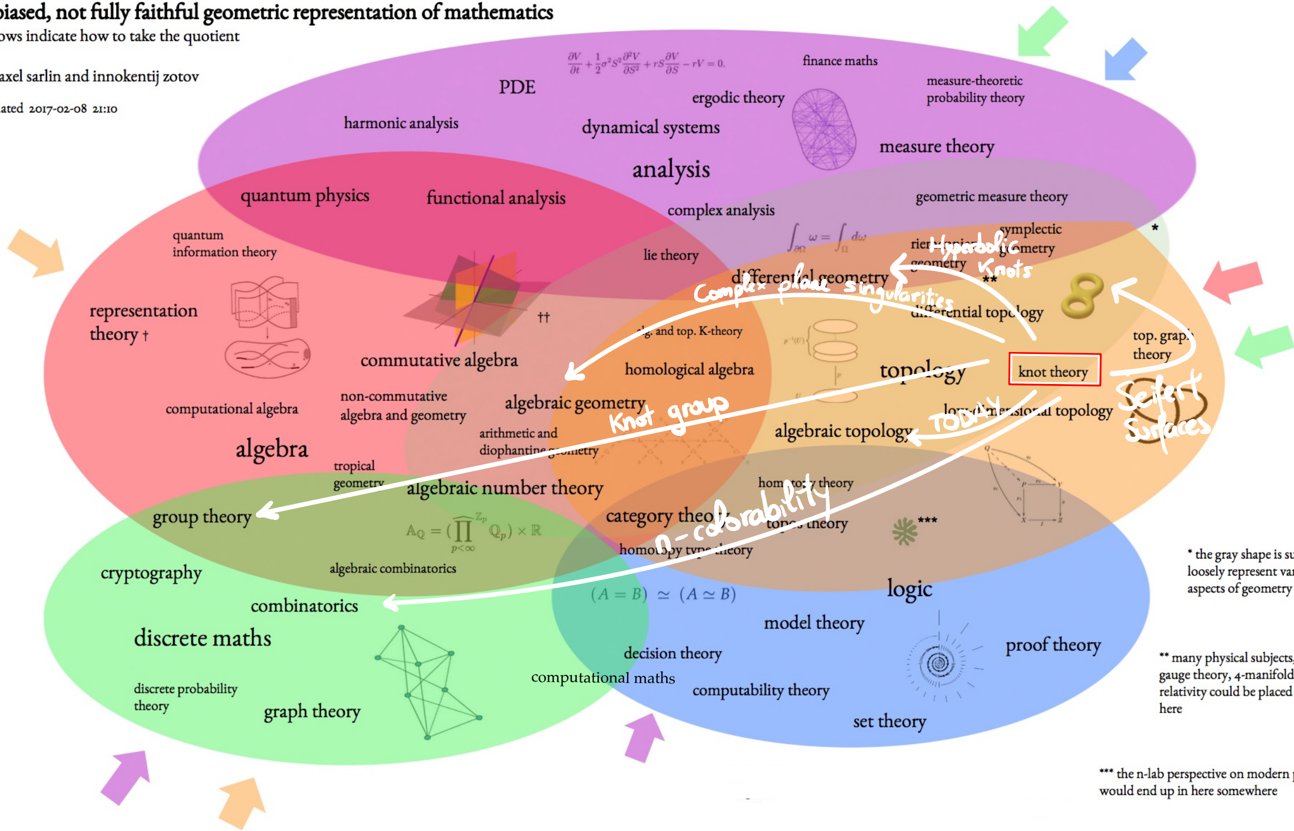
Topological spaces are the most general notion where "continuity" is defined.

a biased, not fully faithful geometric representation of mathematics

arrows indicate how to take the quotient

by axel sarlin and innokentij zotov

updated 2017-02-08 21:10



* the gray shape is supposed to loosely represent various aspects of geometry

** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here

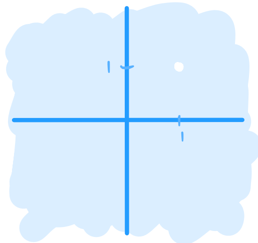
*** the n-lab perspective on modern physics would end up in here somewhere

14. The fundamental group of a space

Idea: take a space X and fix a point $x_0 \in X$. Then

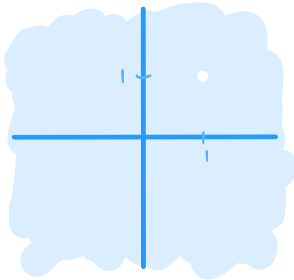
$$\pi_1(X) = \{ \text{paths on } X \text{ starting and ending at } x_0 \text{, up to homotopy} \}$$

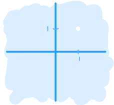
To illustrate this, take $X = \mathbb{R}^2 \setminus \{(1,1)\}$:



Path, homotopy: [Desmos]

Observation: we can compose paths!



In fact, $\pi_1\Big($  $\Big)$ is a group!

Associativity easy, identity element + inverses: in the exercises.

<u>Fact:</u> let X be any topological space. Then $\pi_1(X)$ is a group, called the

Fundamental group of X

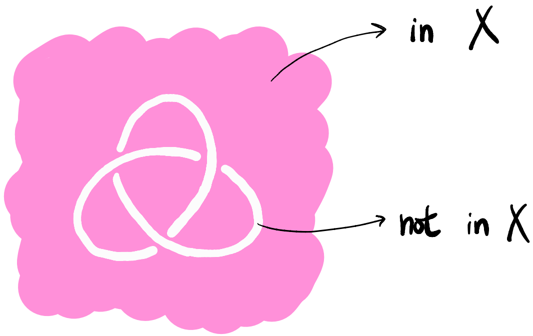
(the beginning of algebraic topology)

So today we defined

- $\pi(K)$ for a link K
- $\pi_1(X)$ for a space X

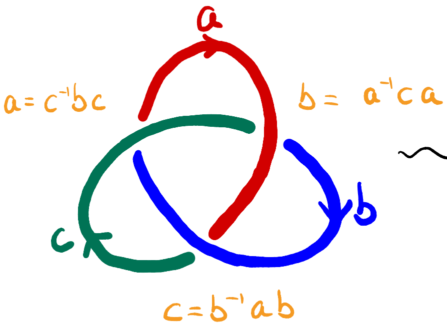
the same?

Almost: given K, consider $\mathbb{R}^3 \setminus K$:



Then $\pi(K) = \pi_1(\mathbb{R}^3 \setminus K)$

But our definition didn't mention paths...

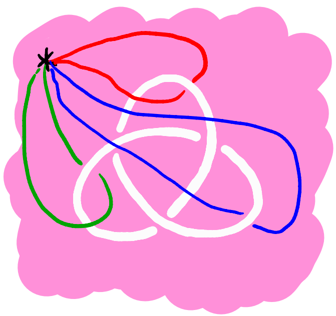


$$\rightsquigarrow \pi(\text{trefoil}) = \langle a, b, c \mid a = c^{-1}bc,\ b = a^{-1}ca,\ c = b^{-1}ab \rangle$$

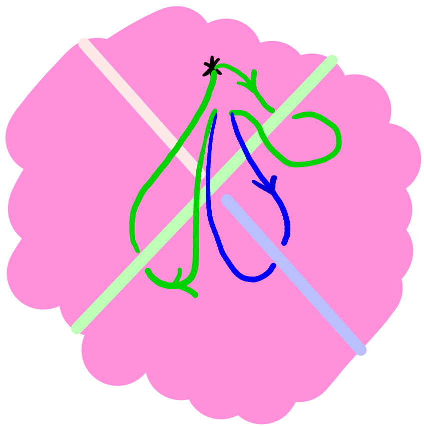
Our weird definition came from :

Generators :

one per arc



Relations :



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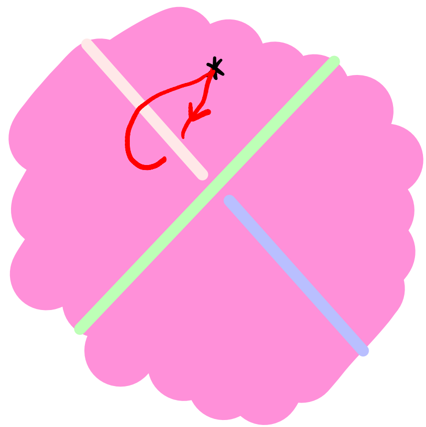
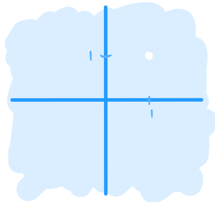


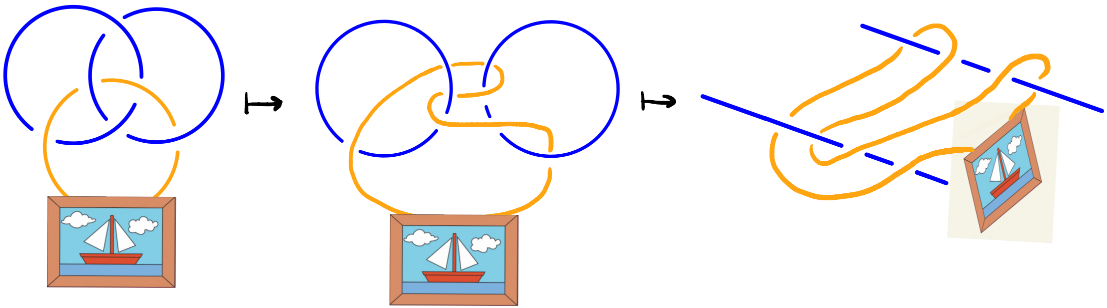
Exercises: explore $\pi_1\Big($  $\Big)$

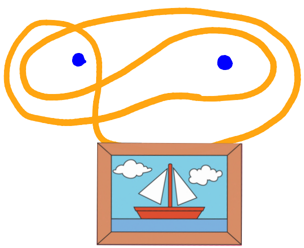
and other fundamental groups

15. A party trick: Brunnian links

Q: How to make the picture fall? (Draw problem)

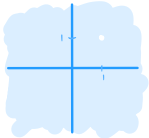
Explanation: Borromean link!

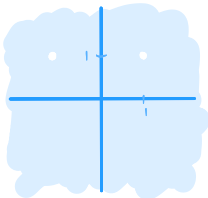




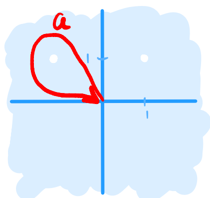
Q: How to generalize this trick?

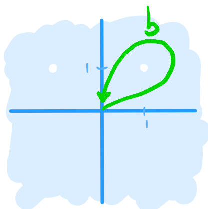
A: Fundamental groups! Let's solve the 2 pin case first

Recall $\pi_1\Big($  $\Big)$

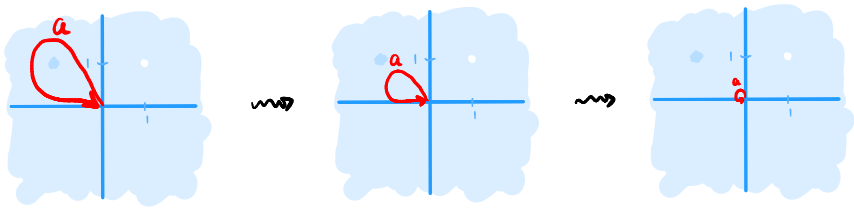
Now consider $X =$ 

We have two generators:





"Removing a pin" corresponds to adding a relation:



$\Rightarrow$ $a = 1$

Similarly, removing the other pin gives $b = 1$.

So we're looking for a word $\neq 1$ in a, b such that:

- If we set $a = 1$, the word simplifies to 1.
- If we set $b = 1$, the word simplifies to 1.

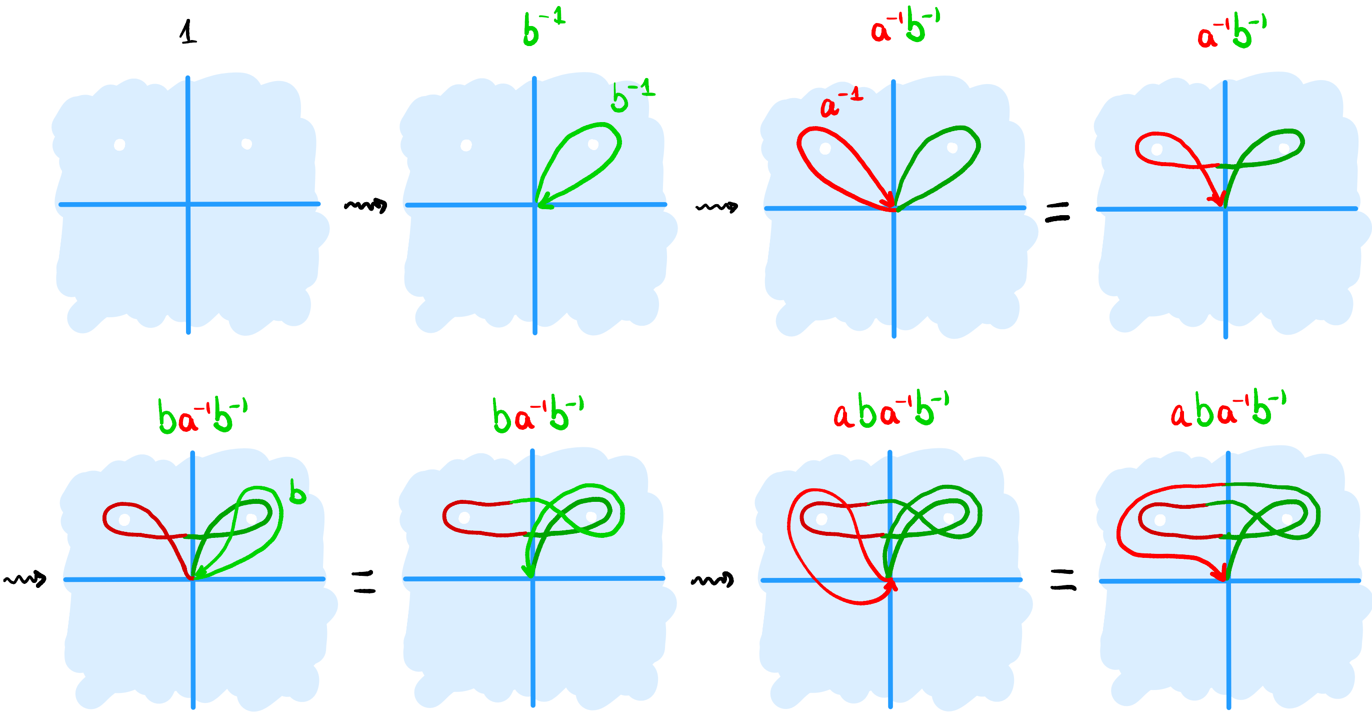
Guesses?

Enter the commutator: $[a, b] = aba^{-1}b^{-1}$

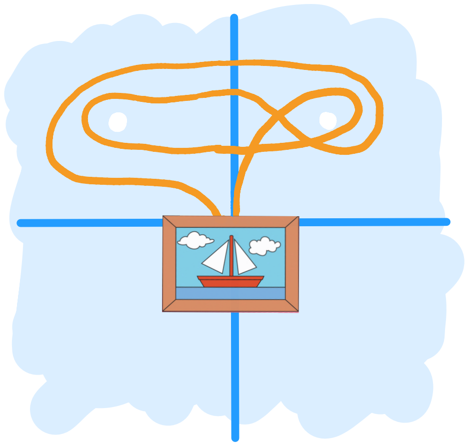
$$[a,b] = aba^{-1}b^{-1} \xrightarrow{a=1} 1b1^{-1}b^{-1} = 1$$

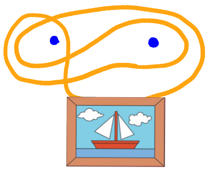
$$[a,b] = aba^{-1}b^{-1} \xrightarrow{b=1} a1a^{-1}1^{-1} = 1$$

Picture:



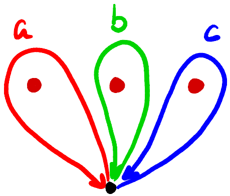
Upshot:



Remark: Same as  , but flipped.

The point: this approach generalizes:

Three pins:



$$[[a,b],c] = [a,b]\,c\,[a,b]^{-1}c^{-1} = a b a^{-1} b^{-1} c\, b a b^{-1} a^{-1} c^{-1}$$

This works:

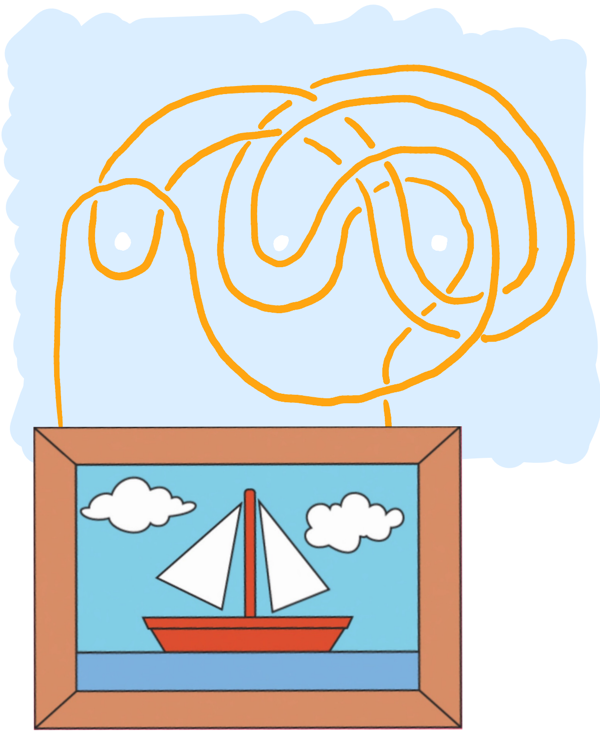
$$[[a,b],c] \xrightarrow{a=1} [[1,b],c] = [1,c] = 1$$

$$[[a,b],c] \xrightarrow{b=1} [[a,1],c] = [1,c] = 1$$

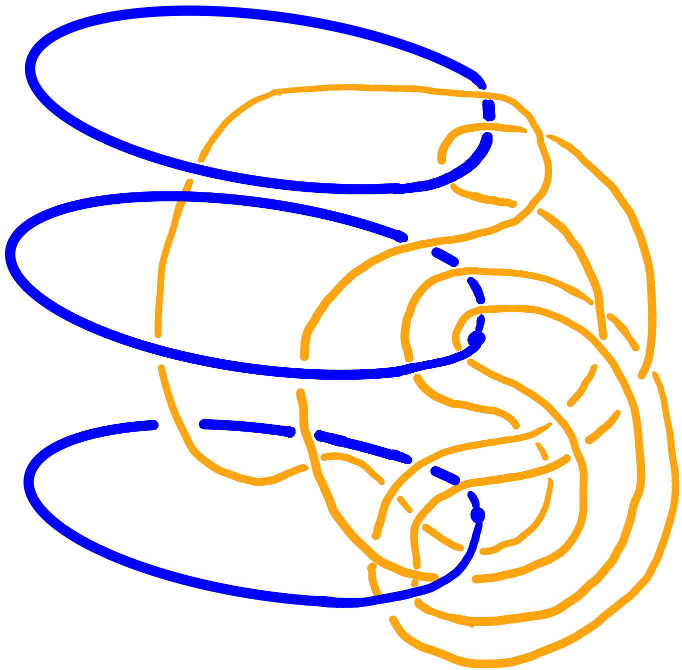
$$[[a,b],c] \xrightarrow{c=1} [[a,b],1] = 1$$

Q?

How this would look like:



... and back to links:



Brunnian link with 4 components

Poll then exercises