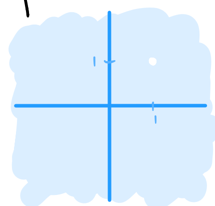
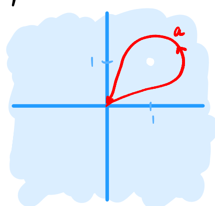
14. The fundamental group of a space.

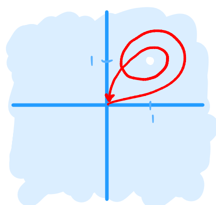
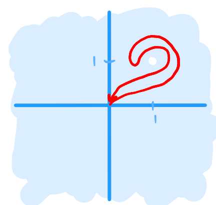
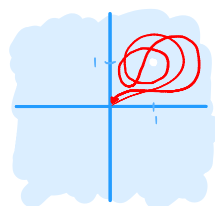
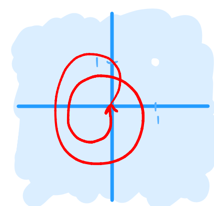
1. Consider again the space $X =$ 

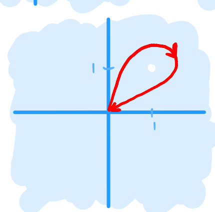
a) Identify the identity element $1 \in \pi(X)$ (i.e. a path such that for any other path x, $x \circ 1 = 1 \circ x = x$)

Consider the path 

, and call it a. Denote by a^n the concatenation of a with itself n times, and write $a^0 = 1$.

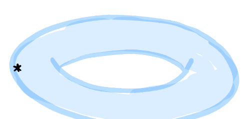
b) Express the following paths as powers of a:

c) What should the path  be in terms of a?

d) Given any path $x \in X$, what should x^{-1} be?

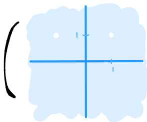
e) Convince yourselves that every element in $\pi_1(X)$ is in fact a power of a.

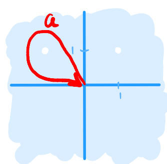
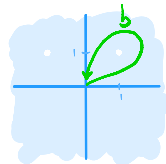
2. Let X be the torus 

. Find two distinct paths a, b based at $*$ such that $ab = ba$.

3. (Challenge) Find a space X such that $\pi_1(X) \cong C_2$.

15. Application: Brunnian links

1. You have 2 pins, A and B. Find a way to hang a picture subject to the following conditions:
    1. If you remove pin A, the picture falls. If you remove pin B, the picture stays up.
    2. If you remove pin A, the picture stays up. If you remove pin B, the picture stays up.
    3. If you remove pin B, the picture falls. If you remove pin A, the picture stays up.
2. Write your solutions from Exercise 1 in terms of the generators of $\pi_1\Big($  $\Big)$, where

   a and b are:  

   $[x,y] = xyx^{-1}y^{-1}$

3. Prove that the following identities hold in the free group on the letters a, b, c.
    1. $(ab)^{-1} = b^{-1}a^{-1}$ (Hint: show that the RHS is the unique element x such that $(ab)x = 1$ and $x(ab) = 1$)
    2. $[a,b] = [a, ba]$
    3. $[a,b][b,c] = [aba^{-1}, ca^{-1}]$ (Hint: $(aba^{-1})^{-1} = ab^{-1}a^{-1}$)
4. Recall the solutions for the 2-pin and 3-pin problem:

   $[a,b] = aba^{-1}b^{-1} \xrightarrow{a=1} [1,b] = 1$

   $[a,b] = aba^{-1}b^{-1} \xrightarrow{b=1} [a,1] = 1$

   $[[a,b],c] \xrightarrow{a=1} [[1,b],c] = [1,c] = 1$

   $[[a,b],c] \xrightarrow{b=1} [[a,1],c] = [1,c] = 1$

   $[[a,b],c] \xrightarrow{c=1} [[a,b],1] = 1$

   Find a solution to the 4-pin problem, using commutators.

   Can you generalize your solution?
5. (Challenge) Interpret the commutator as an OR statement, and use the 3-commutator from Exercise 4 to solve the 2 out of 4 puzzle. You may use the Sage Math code in the second page to check that the picture does not fall when you only remove 1 pin.

Sage code: (use https://sagecell.sagemath.org)

```
F.<a,b,c,d> = FreeGroup();     → Define the free group
def comm(x,y): return x*y*x^-1*y^-1     → Define the commutator
def comm3(x,y,z): return x*y*z*x^-1*y^-1*z^-1     → Define the 3-commutator

rels=[a,b*c]
G=F/rels     → Choose relations to impose. Here a = 1 and bc=1
f = F.hom(G.gens())     → Define the map f="apply the relations"
word=comm(d,b*c)*a^2     → choose some word to simplify. Here, [d, bc] a^2

f(word)==f(1)     → check if the word simplifies to 1. Here, [d,bc]a^2 = [d,1] 1^2 = 1
```

Evaluate

```
True     → So correctly, it gives True
```