Reminder of knot theory so far:

a biased, not fully faithful geometric representation of mathematics

arrows indicate how to take the quotient

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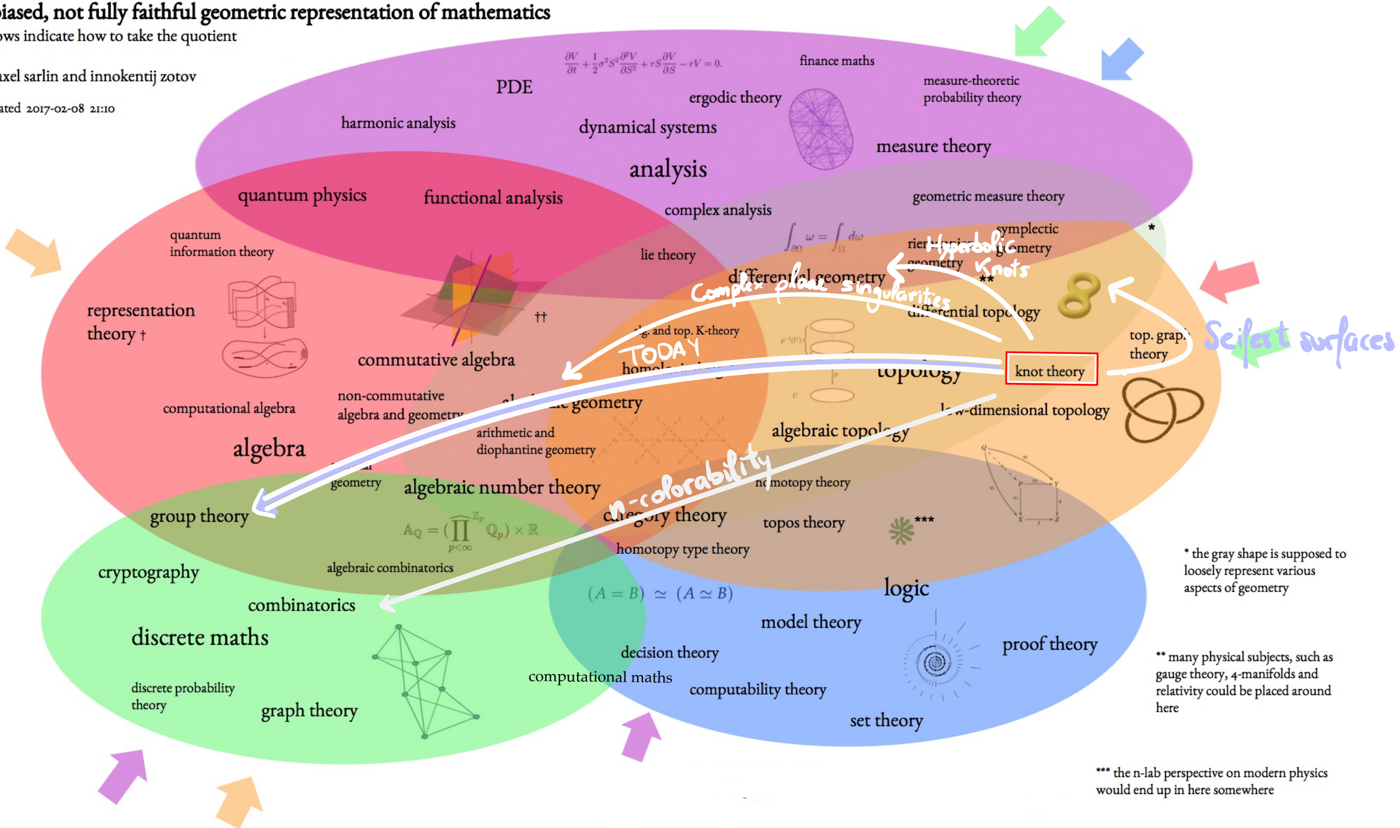


* the gray shape is supposed to loosely represent various aspects of geometry

** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here

*** the n-lab perspective on modern physics would end up in here somewhere

13. Group Theory

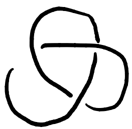
<u>Definition</u>: a set is a collection of objects, without repetitions.

Examples: $\{0,1,2\}$, $\{$ prime knots $\}$, $\{$ real numbers $\}$

finite   infinite   very infinite

The objects inside the sets are called elements, and whenever an element a belongs to a set A, we write $a \in A$.

Examples: $1 \in \{0,1,2,3,\ldots\}$

 $\notin$ $\{$ knots with genus 2 $\}$

Definition: a group is a set G together with an operation $*$ satisfying the following axioms:

- For all $x, y \in G$, $x * y \in G$. Closure
- For all $x, y, z \in G$
$$(x*y)*z = x*(y*z)$$
Associativity
- There exists an element $e \in G$, such that for all $x \in G$,
$$x * e = x \quad \text{and} \quad e * x = x$$
Identity element
- For all $x \in G$ there exists an element $y \in G$ such that
$$x * y = e \quad \text{and} \quad y * x = e$$
Inverse

We write it x^{-1}

Q?

This generalizes many notions you already know:

- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ with $* = +$

  Closure:

  Associativity:

  Unit element:

  Inverses:

- $\mathbb{R}_{>0} = \{$ positive real numbers $\}$ with $* = \cdot$

  Closure:

  Associativity:

  Unit element:

  Inverses:

Q?

- {True, False} with $*$ = XOR

  Closure:

  Associativity:

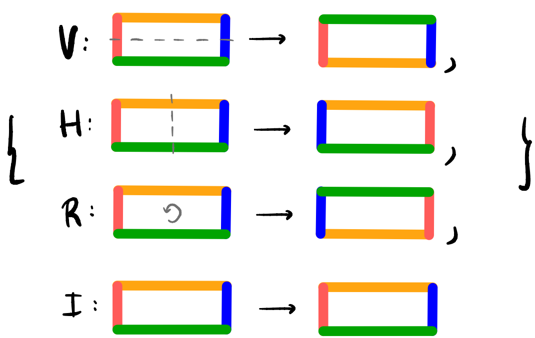
  Unit element:

  Inverses:

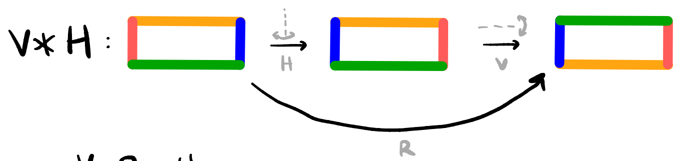
XOR	F	T
F	F	T
T	T	F

"Operation table"

- Symmetries of a rectangle:



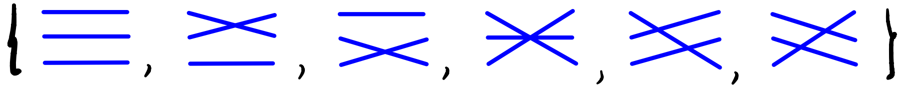
, $*$ = Composition of symmetries:

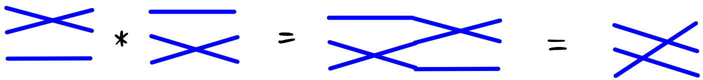


so $V*R = H$

Q?

- Symmetric group on 3 strands, $*$ = concatenation: "S_3"



Example: 

Closure: 6 possibilities, all drawn

Associativity:

(picture)

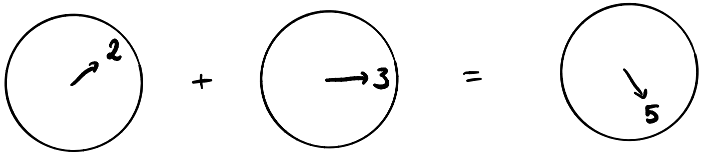
Unit element:

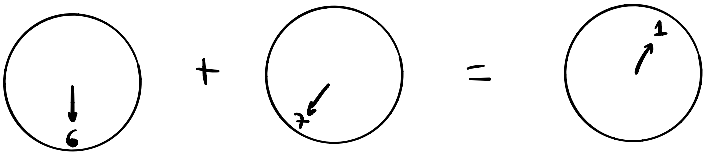
Inverses:

(reverse diagram)

- Cyclic group with 12 elements : $\mathbb{Z}_{12}$

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $\quad * = + \pmod{12}$





Closure:

Associativity :

Unit element :

Inverses:

Some nonexamples:

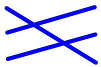
- $\mathbb{Z}$, $* = -$

  Closure:

  Associativity: Q

  Unit element:

  Inverses:

- $\{$  ,  $\}$

  Q: what fails here?

Groups by generators and relations

Definition: The free group on 2 letters is the set $G = \{$ words formed by $a, b, a^{-1}, b^{-1} \}$ and $* =$ concatenation.

Example: $aaba^{-1}b * b^{-1}a^{-1}b = aaba^{-1}bb^{-1}a^{-1}b$

$= aaba^{-1}a^{-1}b$

$= aaba^{-1}a^{-1}b$

$= a^2ba^{-2}b$

This forms a(n infinite) group:

Closure:

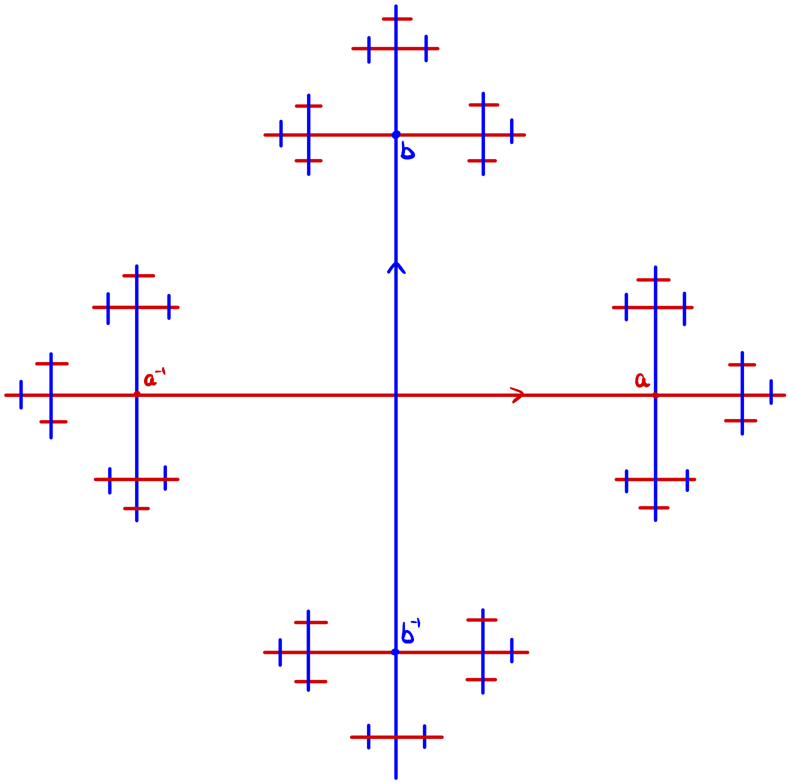
Associativity:

Identity element:

Inverses:

Remark: Similarly, we can define the free group on n letters.

Picture of F_2:



Groups by generators and relations

We can define new groups by imposing "relations" on the free group:

Example: $$G = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{ab = ba,\ a^2 = 1,\ b^2 = 1}_{\text{relations}} \rangle$$

This is again the group of words a, b, a^{-1}, b^{-1}, ab, ba, $a^{-1}b$, ab^2, $ba^2b^{-1}ab^2$, ...

... but now $a^2 =$

$a^3 =$

$a^{-1} =$

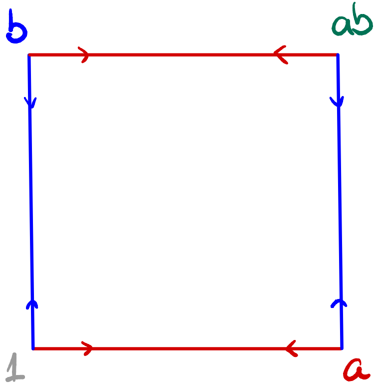
$a^n =$

$b^n =$

$a^2b^{-1}a^{-2}b^3 =$

Poll: how many elements does G have? 2, 4, 10, or ∞?

<u>Aside</u>: picture of $\langle a, b \mid ab = ba,\ a^2 = 1,\ b^2 = 1 \rangle$

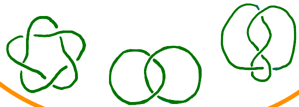


<u>Exercises</u>: investigate these examples, and more

14. The knot group

We will be assigning

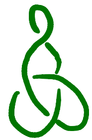
Links



$\rightsquigarrow$

Groups

C_n $\quad S_n \quad$ $\mathbb{Z}$ $\quad \mathrm{Sym}(\square)$

Just like we had a notion of equivalence of links:  $\cong$ 

we have a notion of **isomorphism** of groups

For example, take $G = (\{1,-1\}, \cdot)$ and $H = (\{\text{True}, \text{False}\}, \text{XOR})$

	1	-1
1	1	-1
-1	-1	1

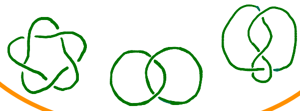
$\xleftrightarrow{\cong}$ "the same"

	T	F
T	T	F
F	F	T

Back to Knots

Recall that we seek

Links



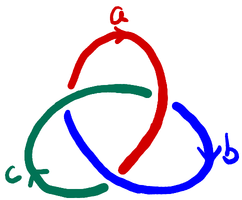
$\rightsquigarrow$

Groups

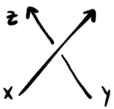
C_n S_n $\mathbb{Z}$ $\mathrm{Sym}(\square)$

Definition: *Knot group* of an oriented link with diagram D is the group given by:

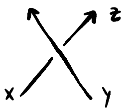
- Generators: arcs in D



- Relations: for each crossing



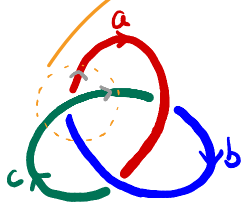
$\Rightarrow \quad z = x^{-1} y x$

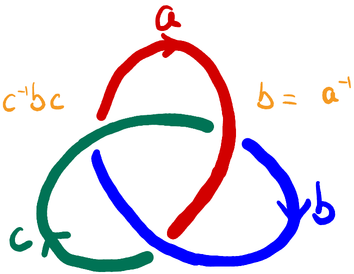


$\Rightarrow \quad z = x y x^{-1}$

so

$a = c^{-1} b c$





$a = c^{-1}bc$ $\qquad$ $b = a^{-1}ca$ $\qquad$ $c = b^{-1}ab$

$$\rightsquigarrow \pi(\text{trefoil}) = \langle a, b, c \mid a = c^{-1}bc,\ b = a^{-1}ca,\ c = b^{-1}ab \rangle$$

<u>Fundamental group of the link</u>

Notice: c can be written in terms of a and b

Relations become:
$$a = (b^{-1}ab)^{-1}\, b\, (b^{-1}ab) = b^{-1}a^{-1}bab$$
$$b = a^{-1}(b^{-1}ab)a = a^{-1}b^{-1}aba$$

"Elimination"

Equivalently, $aba = bab$ (see Exercises)

Thus $\boxed{\pi(\text{trefoil}) = \langle a, b \mid aba = bab \rangle}$

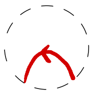
Theorem: Up to isomorphism, the knot group is an invariant of links

In other words, two diagrams coming from the same link give isomorphic groups.

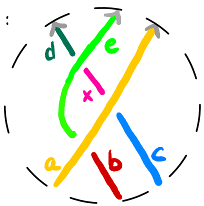
Proof: We check invariance under the Reidemeister moves:

RI: 

Relation: $a = a^{-1}aa$



Relation: ( )

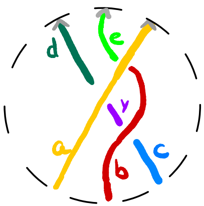
RIII: 

Relations: $\begin{cases} e = a^{-1}ba \\ x = a^{-1}ca \\ d = e^{-1}xe \end{cases}$

Eliminate x: $\begin{cases} e = a^{-1}ba \\ a^{-1}ca = ede^{-1} \end{cases}$

Eliminate e: $a^{-1}ca = a^{-1}bada^{-1}b^{-1}a$

$$\boxed{c = bada^{-1}b^{-1}}$$



Relations: $\begin{cases} e = a^{-1}ba \\ y = b^{-1}cb \\ d = a^{-1}ya \end{cases}$

Eliminate y: $\begin{cases} e = a^{-1}ba \\ b^{-1}cb = ada^{-1} \end{cases}$

Eliminate e: $b^{-1}cb =$ [illegible]a^{-1}

$$\boxed{c = bada^{-1}b^{-1}}$$

RII: You will prove it in the exercises.

$\square$

Exercises: compute and explore knot groups