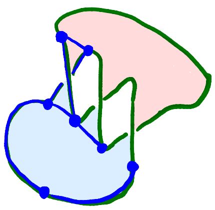
7. Seifert surfaces

1. Applying the Seifert algorithm to a certain diagram for the trefoil, we get the following surface:

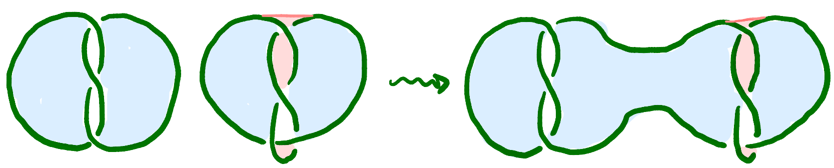


- Complete the deflated polygon of the given surface in order to compute its Euler characteristic
- Use this to prove that $g(\text{trefoil}) = 1$.

2. Use the method you used in 1 to prove that the Euler characteristic of the Seifert surface obtained from applying the Seifert algorithm with s Seifert circles and c crossings is given by $\chi = s - c$. If the link is in fact a knot, give a formula for the genus in terms of s and c.

3. Let K_1, K_2 be knots. Starting from Seifert surfaces for each of K_1, K_2, construct a Seifert surface for $K_1 \# K_2$. Deduce that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.

Suggestive picture:

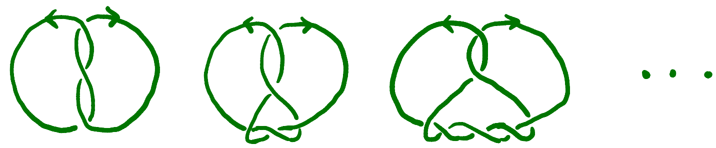


4. Is it true that $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$?

8. Applications: no-untying and knot primality

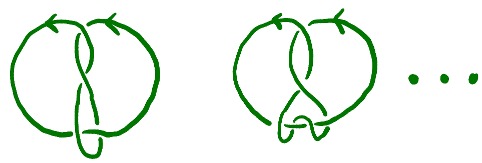
1. Prove that a knot of genus 1 is prime.

2. Use your result in the previous section ($\chi = s - c$) to show that the genus of each of the following knots is 1:



Using 1, conclude that there exist infinitely many prime knots.

3. (Optional): Prove that the knots



also have genus 1.

4. (Harder) The crossing number of a knot K is the minimum number $cr(K)$ of crossings needed to draw a link diagram. Prove that there exist knots with arbitrarily large $cr(K)$.

5. (Open question) Prove that cr is additive, i.e. $cr(K_1 \# K_2) = cr(K_1) + cr(K_2)$.