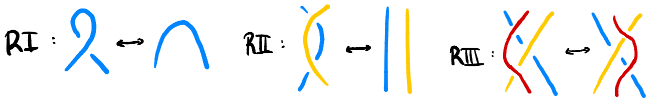
Reminder:

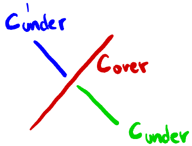
- Knots and links: 

- Showing they're equal: Reidemeister moves



- Showing they're different: invariants

  ↳ n-colorability



$$2c_{over} \equiv c'_{under} + c_{under} \pmod n$$

Today: higher dimensions

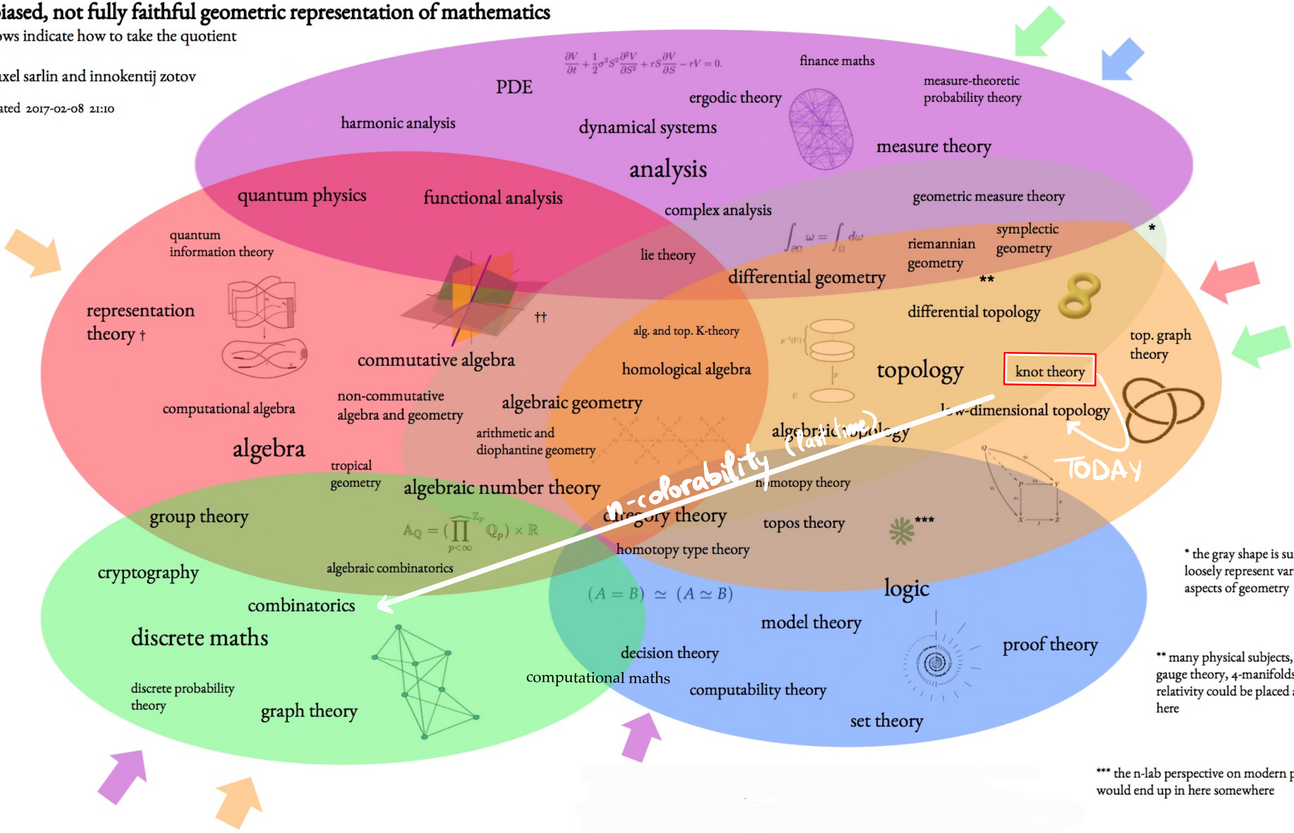
Today we will investigate

a biased, not fully faithful geometric representation of mathematics

arrows indicate how to take the quotient

by axel sarlin and innokentij zotov

updated 2017-02-08 21:10



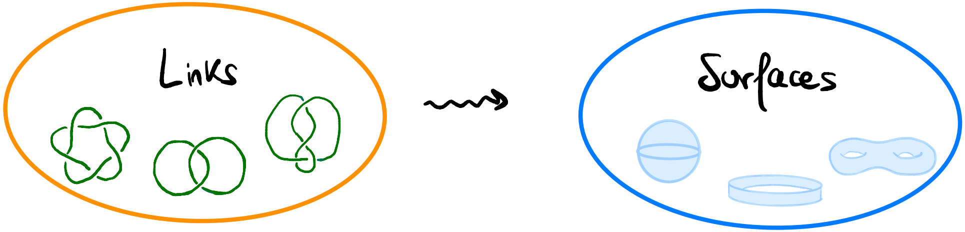
* the gray shape is supposed to loosely represent various aspects of geometry

** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here

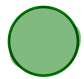
*** the n-lab perspective on modern physics would end up in here somewhere

5. The topology of surfaces

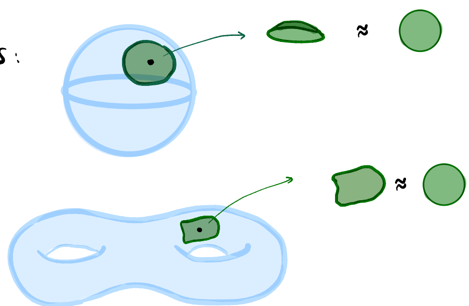
We will associate



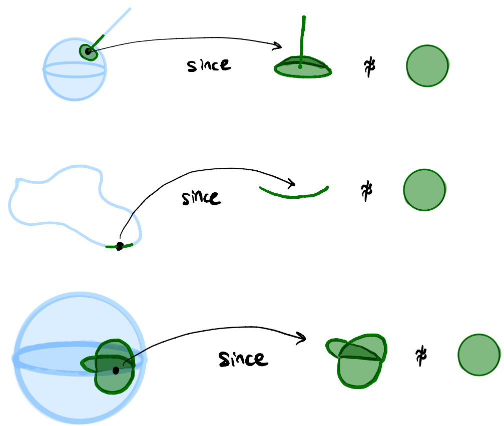
<u>Definition</u>: a surface without boundary is a subset of 3D space such that around every point there is a "disk": a copy of $B^2 = \{x^2+y^2 \le 1\}$ =



Examples:



Nonexamples:

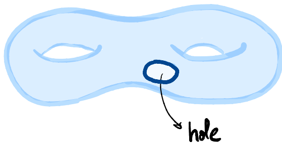


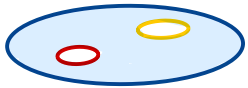
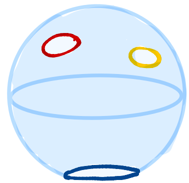
"Local condition"

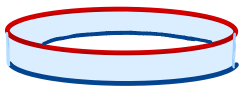
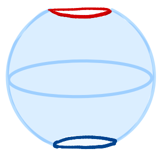
Q?

Definition: a surface **with boundary** is the space obtained by removing one or more disks from a surface without boundary

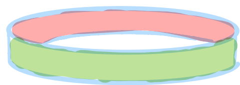
Examples:



 $\approx$ 

 $\approx$ 

Warning: we will not consider non-orientable surfaces, that is, surfaces that cannot be painted with two colors:


orientable


non-orientable

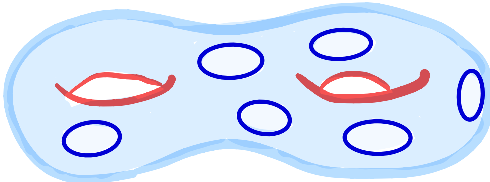
Q?

Fact: a surface is determined by

number of "donut holes" genus g

number of removed discs #boundary components b

Example:

$T_{g,b} =$ 

is the surface with genus $g = 2$

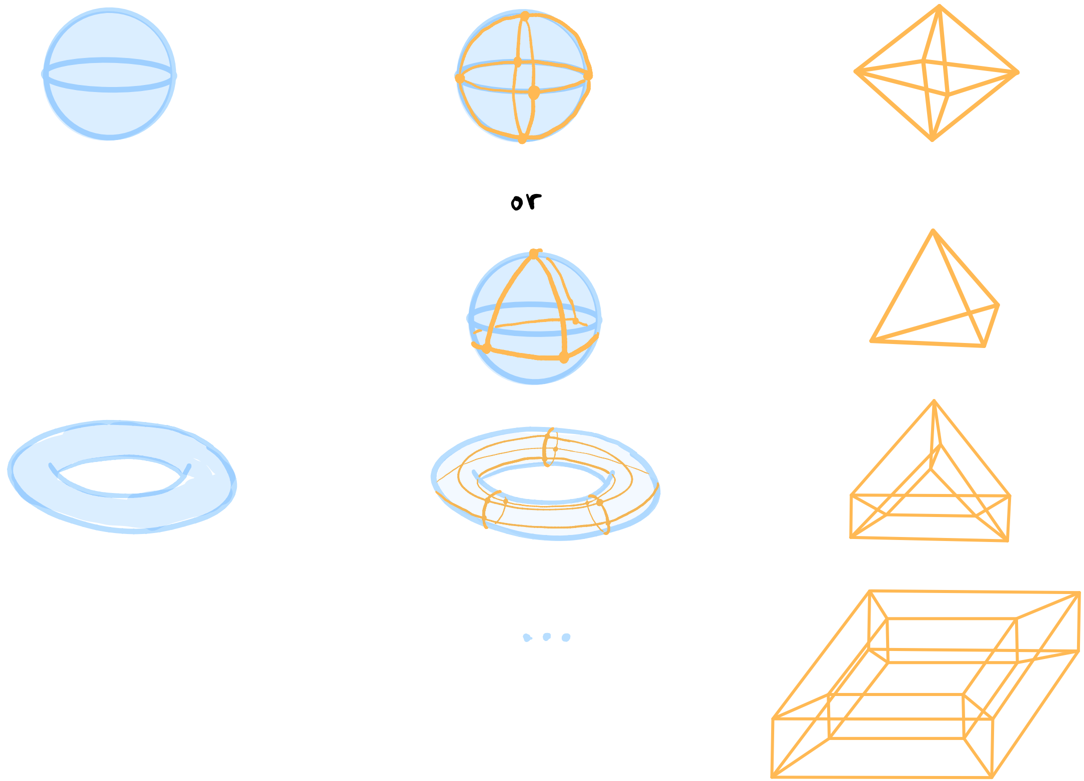
and #bdary components $b = 6$

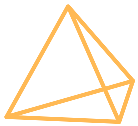
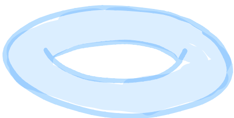
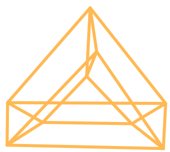
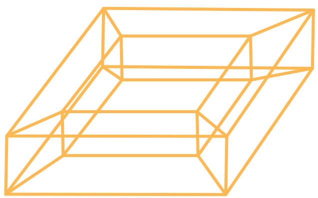
Technical remark: just like with knots, we may stretch surfaces without tearing them apart – this does not change the surface.

Q?

The Euler characteristic of a surface

Surface ⇝ "Inflated" Polyhedron ⇝ Polyhedron (aka polygon mesh)



Surface	$\rightsquigarrow$	Polyhedron (aka polygon mesh)	V	E	F	χ
			6	12	8	
			4	6	4	
			9	18	9	
			16	32	16	

Q?

The Euler characteristic of a surface S is

$$\chi(S) = V - E + F$$

for any deflated polyhedron for S with

$V =$ # vertices

$E =$ # edges

$F =$ # faces

Theorem (Legendre, 1700s): the Euler characteristic is independent of the choice of polyhedron.

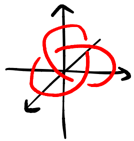
Your first task today: Conjecture and prove a formula for $\chi(T_{g,b})$

Q? Exercises

6. Topology in higher dimensions : movies

Fundamental idea in mathematics: <u>Generalize</u>

So far: A Knot is a "circle" inside $\mathbb{R}^3 = \{(x,y,z)\}$



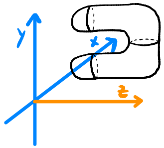
One step up: a "sphere" inside $\mathbb{R}^4 = \{(x,y,z,t)\}$??

Problem: How do you even draw this? How can I picture 4 dimensions?

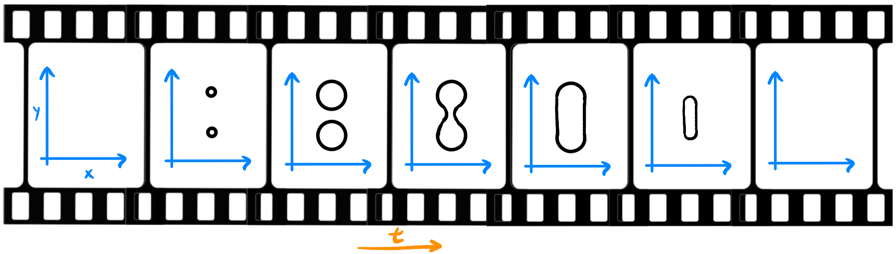
One possibility: represent the 4th variable as time.

To see how this would work, let us represent a 2D surface as a "movie"

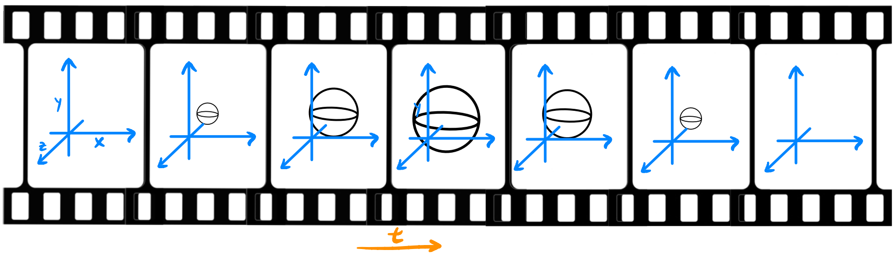
Spatial representation:



Movie representation:



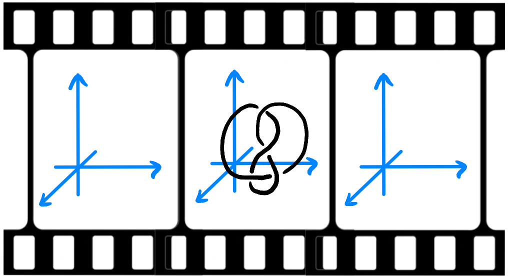
Now the following movie is a representation of the 3-sphere in $\mathbb{R}^4$:



Question: why not study 1-dimensional knots in $\mathbb{R}^4$?

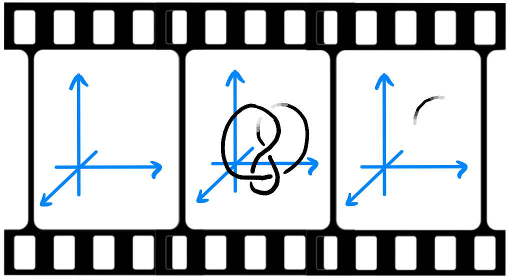
Answer: They all untangle!

To see this, place the knot in $\mathbb{R}^4$ and bring the whole thing to $t=0$.



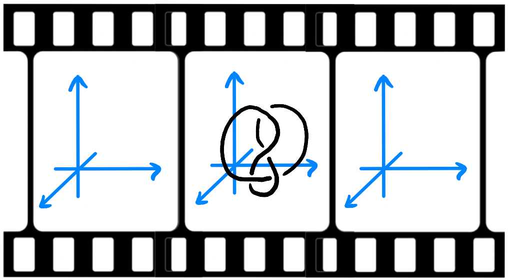
$t=0$

Now, move the understrand slightly into $t=1$



$t=0$ $t=1$

Finally, bring back the strand so that it becomes the undercrossing:

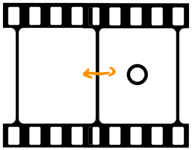
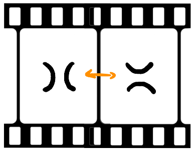


$t=0$

This is the unknot!

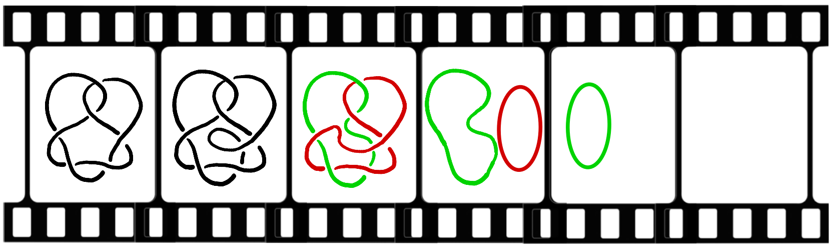
A recent development on slice knots

A "Knot movie" represents a disk in $\mathbb{R}^4$ if and only if the possible movie moves are

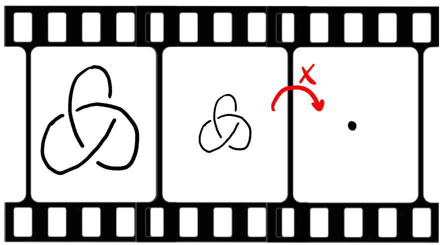
 or , representing  and , or Reidemeister moves.

A knot is called "slice" if it can be obtained as one of the frames in such a movie.
(So that it is a "slice" a phere)

Example:



Nonexample:

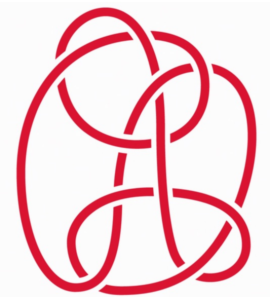


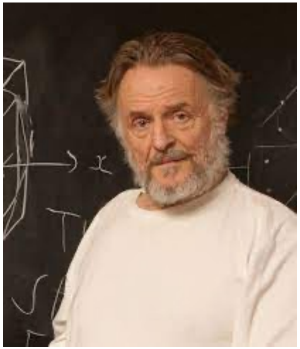
Two ways to do this — Topologically slice knot
— Smoothly slice knot (more restrictive)

Out of the thousands of knots with ≤ 12 crossings, mathematicians showed that

topologically slice $\longleftrightarrow$ smoothly slice

for all but one knot, Conway's knot:





John Conway (1937-2020)

Lisa Piccirillo (2020):

<u>Theorem</u>: The Conway knot is not smoothly slice.

(The proof uses a sophisticated invariant called Rasmussen's s-invariant).