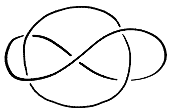
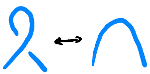
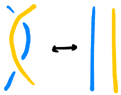
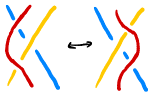
Reminder of last time

- We defined knots and links



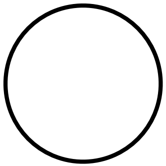
- We saw elementary tranformations to manipulate link diagrams

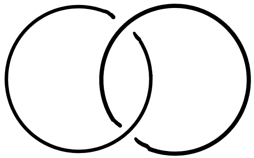
RI :  RII :  RIII : 

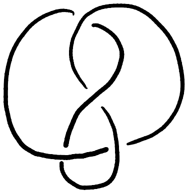
- We saw how to input links into Sage

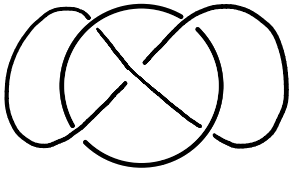
3. Combinatorics and our first invariant

<u>Question</u>: Which of the following links are you able to tell apart?

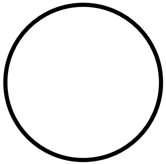
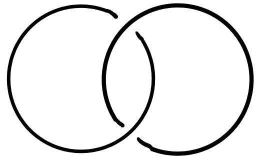
$L_0 =$ 

$L_1 =$ 

$L_2 =$ 

$L_3 =$ 

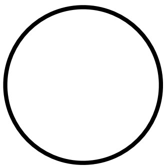
Idea:

 $\neq$ 

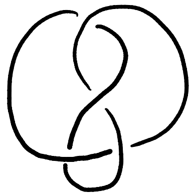
1 connected component

2 connected components

Now



?



?

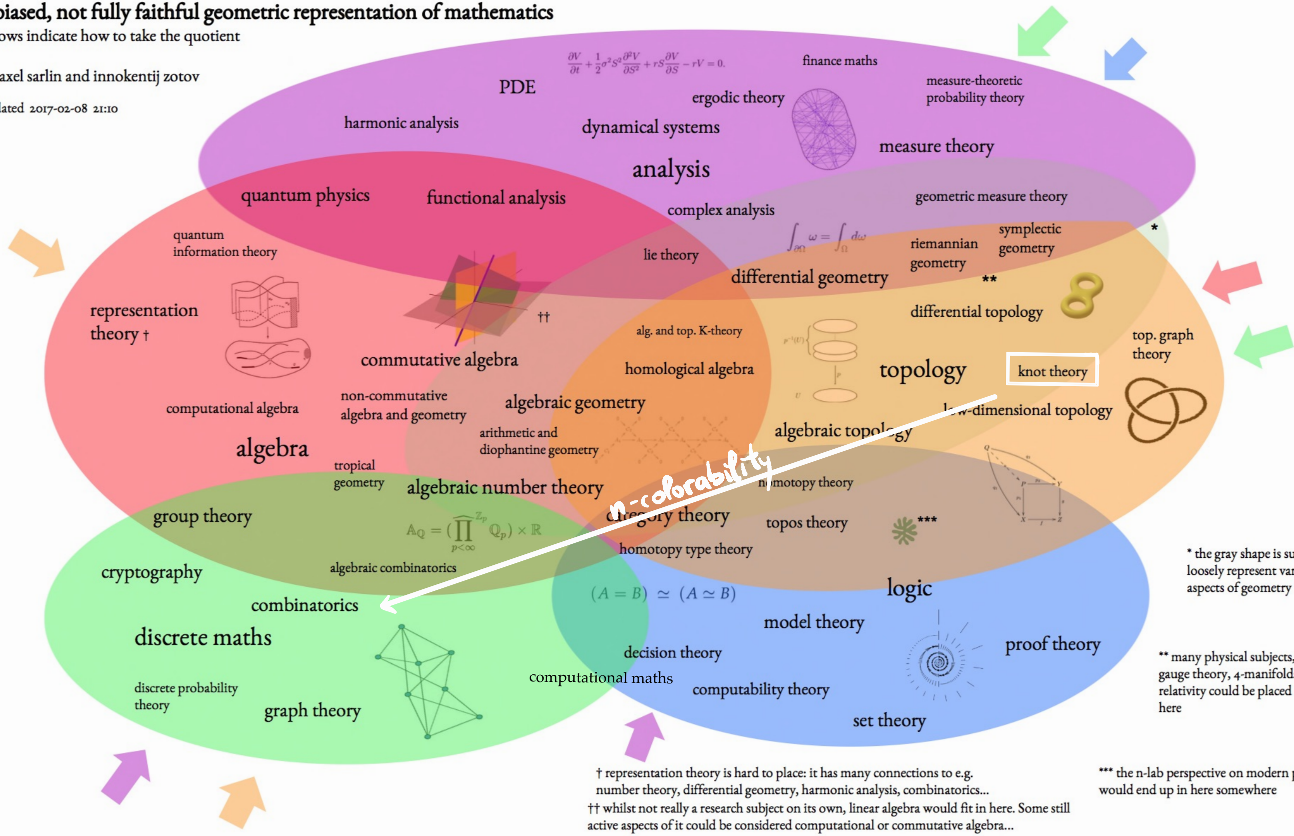
Today: a combinatorial invariant

a biased, not fully faithful geometric representation of mathematics

arrows indicate how to take the quotient

by axel sarlin and innokentij zotov

updated 2017-02-08 21:10



* the gray shape is supposed to loosely represent various aspects of geometry

** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here

† representation theory is hard to place: it has many connections to e.g. number theory, differential geometry, harmonic analysis, combinatorics...

†† whilst not really a research subject on its own, linear algebra would fit in here. Some still active aspects of it could be considered computational or commutative algebra...

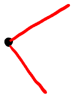
*** the n-lab perspective on modern physics would end up in here somewhere

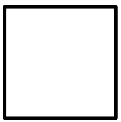
But... what is combinatorics?

The mathematics of <u>counting</u>

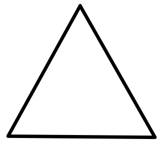
<u>Example</u>

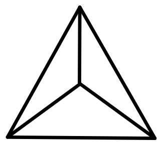
Count in how many ways the edges of the following graph can be colored so that

- We use ≤ 2 colors
-  two adjacent edges cannot have the same color



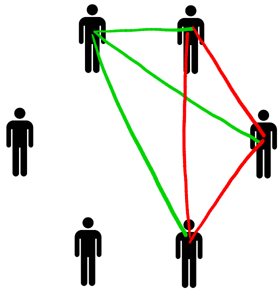
Poll:

1. How many 3-colorings does  have?

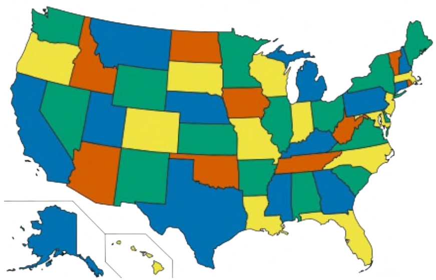
2. How many 3-colorings does  have?

Colorings in math

- Ramsey theory



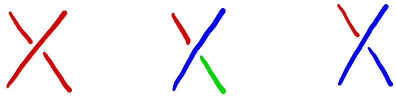
- 4-color theorem



Back to Knots

<u>Definition</u>: A link diagram is 3-colorable iff it is possible to assign colors to its edges so that:

a) At each crossing, either one or three colors are used:

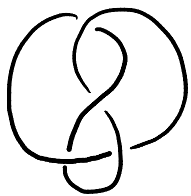


valid   valid   <u>not</u> valid

b) Only using one color everywhere is not allowed

Examples:

Nonexample:



Question: if I have two link diagrams for the same knot, can one be 3-colorable but not the other one? E.g.

<u>Theorem</u> : tricolorability is a link invariant.

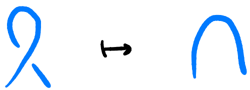
(If L is a link and D_1, D_2 are diagrams for it, then D_1 is tricolorable $\iff$ D_2 is tricolorable)

<u>Proof</u>: A) • Suppose D_1 is 3-colorable

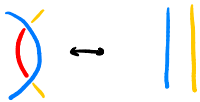
• Reidemeister moves $D_1 \xrightarrow{R?} D_1' \to \dots \to D_2$

• Coloring for D_1' :

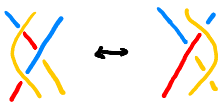
↳ RI :



↳ RII :
(one color: easy)



↳ RIII :



• Repeat for D_1'', D_1''', ..., D_2 $\Rightarrow$ D_2 is tricolorable

B) • Suppose D_1 is <u>not</u> tricolorable. Then if D_2 is tricolorable then D_1 is, a contradiction.

$\Rightarrow$ D_2 is not tricolorable.

Q, Install SnapPy

4. Modular arithmetic and n-colorability

Arithmetic modulo n

"Clock arithmetic". Fix an integer $n \geq 1$.

Definition: Two integers a, b are congruent modulo n if they have the same remainder when divided by n.

We write $a \equiv b \pmod{n}$

Examples: 17 divided by 12 gives a remainder of 5

$\Rightarrow \quad 17 \equiv 5 \pmod{12}$

$25 = 12 \cdot 2 + 1$

$13 = 12 \cdot 1 + 1$

$\Rightarrow \quad 25 \equiv 13 \equiv 1 \pmod{12}$

Also negative: $-1 = 12 \cdot (-1) + 11$

$\Rightarrow \quad -1 \equiv 11 \pmod{12}$

We can do arithmetic modulo n:

Take $n = 7$.

$$5 + 4 = 9 \equiv 2 \pmod 7$$

$$3 - 5 = -2 \equiv 5 \pmod 7$$

$$2 \cdot 4 = 8 \equiv 1 \pmod 7$$

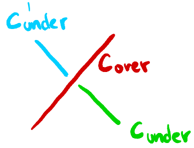
Note: 2 and 4 are multiplicative inverses modulo 7.

n- colorability

Definition: let $n \geq 1$.

A link diagram L is n-colorable if we can assign a remainder in $\{0, 1, \dots, n-2, n-1\}$ so that:

- At each crossing



$$2\,C_{over} \equiv C'_{under} + C_{under} \pmod n$$

- More than one color is used.

Remark: This definition coincides with 3-colorability for $n=3$:

$$2\,C_{over} \equiv C'_{under} + C_{under}$$

$C_{over} \equiv 0 \Rightarrow$ $C'_{under} \equiv C_{under} \equiv 0$ or $C'_{under} \equiv 1 \quad C_{under} \equiv -1$ or $C'_{under} \equiv -1 \quad C_{under} \equiv 1$

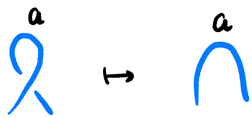
$C_{over} \equiv 1 \Rightarrow$ $C'_{under} \equiv C_{under} \equiv 1$ or $C'_{under} \equiv 2 \quad C_{under} \equiv 0$ or $C'_{under} \equiv 0 \quad C_{under} \equiv 2$

$C_{over} \equiv 2 \Rightarrow$ $C'_{under} \equiv C_{under} \equiv 2$ or $C'_{under} \equiv 1 \quad C_{under} \equiv 0$ or $C'_{under} \equiv 0 \quad C_{under} \equiv 1$

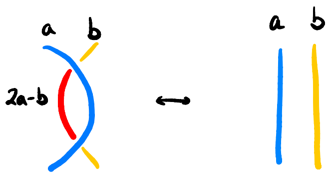
Theorem: n-colorability is a link invariant

Proof: we will show invariance under the Reidemeister moves

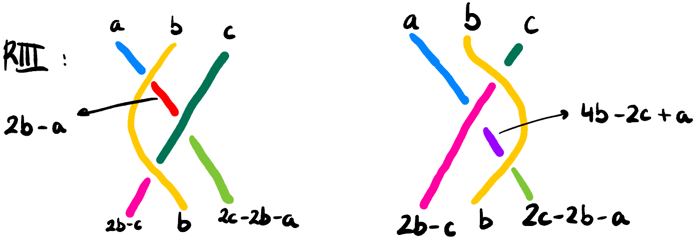
↳ RI :



↳ RII :
(one color: easy)



↳ RIII :



Q? SnapPy