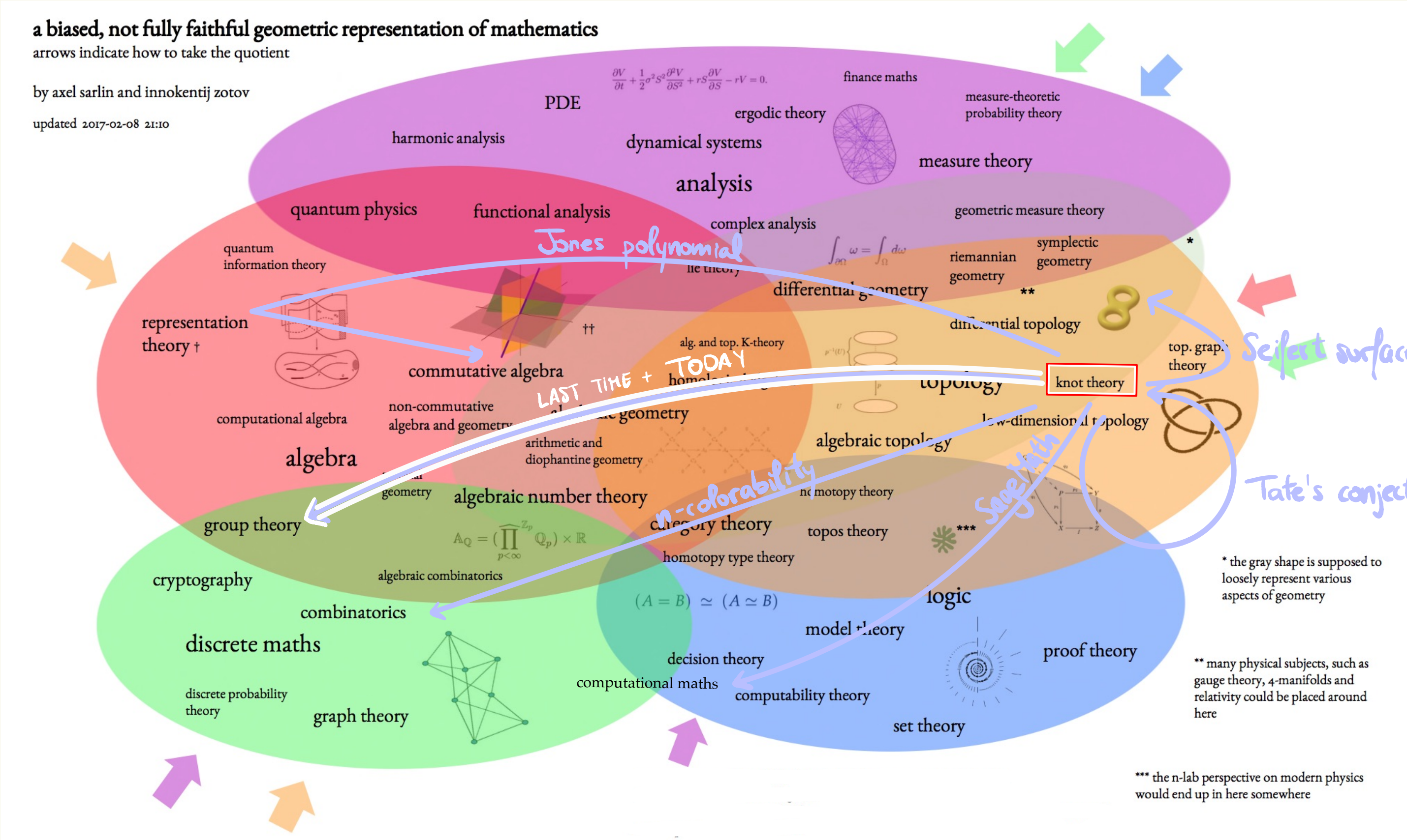
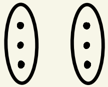
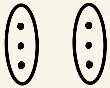
Reminder of knot theory so far:



Reminder of knot theory so far:

- A set is a collection of objects

↳ A function between sets $f: X \to Y$ is an isomorphism of sets if:

- 
- 

- A group is a set G with an operation $*$ such that:
  - $*$ is associative
  - There is an identity element
  - Every element has an inverse

↳ A function between groups $f: G \to H$ is an isomorphism of groups if:

- It is an isomorphism of sets
- It respects the operation: $f(g_1 * g_2) = f(g_1) * f(g_2)$

13. The fundamental group of a knot

Groups by generators and relations

Definition: The free group on 2 letters is the set $G = \{ \text{words formed by } a, b, a^{-1}, b^{-1} \}$ and $* =$ concatenation.

Example: $aaba^{-1}b * b^{-1}a^{-1}b = aaba^{-1}bb^{-1}a^{-1}b$

$= aaba^{-1}a^{-1}b$

$= aaba^{-1}a^{-1}b$

$= a^2ba^{-2}b$

This forms a(n infinite) group:

Closure:

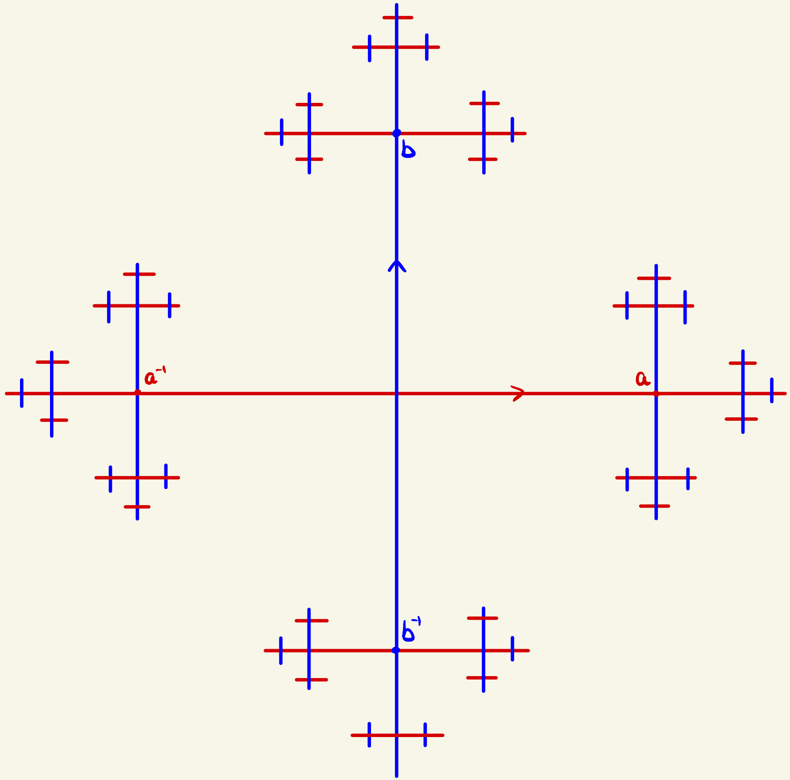
Associativity:

Identity element:

Inverses:

Remark: Similarly, we can define the free group on n letters.

Picture of F_2:



Groups by generators and relations

We can define new groups by imposing "relations" on the free group:

Example: $G = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{ab = ba,\ a^2 = 1,\ b^2 = 1}_{\text{relations}} \rangle$

This is again the group of words $a,\ b,\ a^{-1},\ b^{-1},\ ab,\ ba,\ a^{-1}b,\ ab^2,\ ba^2b^{-1}ab^2, \ldots\ldots$

... but now $a^2 =$

$a^3 =$

$a^{-1} =$

$a^n =$

$b^n =$

$a^2b^{-1}a^{-2}b^3 =$

Poll: how many elements does G have? $2,\ 4,\ 10,$ or ∞?

In fact $\langle a, b \mid ab = ba,\ a^2 = 1,\ b^2 = 1 \rangle \cong C_2 \times C_2$ from last time!

G	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

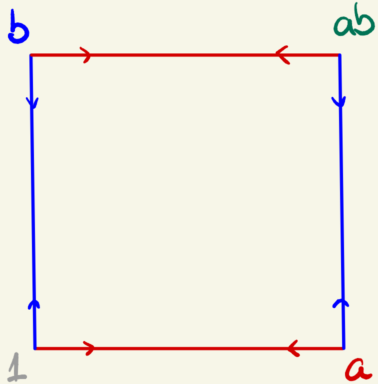
$C_2 \times C_2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

So $f : G \to C_2 \times C_2$

$$1 \mapsto (0,0)$$
$$a \mapsto (1,0)$$
$$b \mapsto (0,1)$$
$$ab \mapsto (1,1)$$

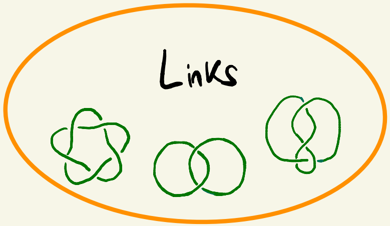
is an isomorphism.

Aside: picture of $\langle a, b \mid ab = ba,\ a^2 = 1,\ b^2 = 1 \rangle$



Back to Knots

Recall that we seek



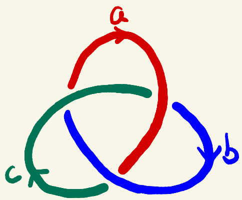
$\rightsquigarrow$

Groups

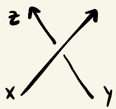
C_n $\quad S_n$ $\quad \mathbb{Z}$ $\quad \mathrm{Sym}(\square)$

Definition: fundamental group of an oriented link with diagram D is the group given by:

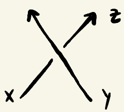
- Generators: arcs in D



- Relations: for each crossing

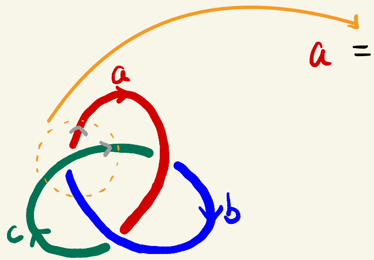


$\Rightarrow \quad z = x^{-1} y x$

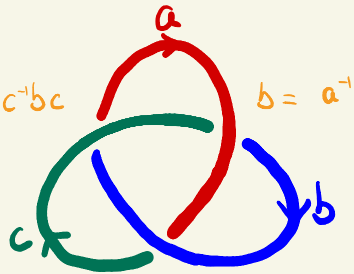


$\Rightarrow \quad z = x y x^{-1}$

so



$a = c^{-1} b c$



$a = c^{-1}bc$ $\quad$ $b = a^{-1}ca$ $\quad$ $c = b^{-1}ab$

$\rightsquigarrow \pi(\text{trefoil}) = \langle a, b, c \mid a = c^{-1}bc,\ b = a^{-1}ca,\ c = b^{-1}ab \rangle$

<u>Fundamental group of the link</u>

Notice: c can be written in terms of a and b

Relations become: $a = (b^{-1}ab)^{-1}\, b\, (b^{-1}ab) = b^{-1}a^{-1}bab$

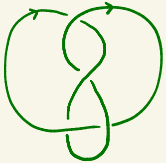
$b = a^{-1}(b^{-1}ab)a = a^{-1}b^{-1}aba$

"Elimination"

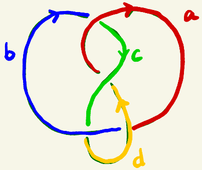
Equivalently, $aba = bab$ (see Exercises)

Thus $\boxed{\pi(\text{trefoil}) = \langle a, b \mid aba = bab \rangle}$

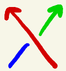
Q

Another example: $K =$ 

Step 1: Generators



Step 2: Relations



$c = a b a^{-1}$

$a = c d c^{-1}$

$d = b^{-1} c b$

$b = d^{-1} a d$

$$\pi(K) = \langle a, b, c, d \mid a = c d c^{-1},\ b = d^{-1} a d,\ c = a b a^{-1},\ d = b^{-1} c b \rangle$$

Step 3: Simplify

$$\pi(K) = \langle a, b, c, d \mid a = c\,d\,c^{-1},\quad b = d^{-1} a\, d,\quad c = a\,b\,a^{-1},\quad d = b^{-1} c\, b \rangle$$

$$= \langle a, b, d \mid a = \underline{a\,b\,a^{-1}}\,d\,\underline{a\,b^{-1}a^{-1}},\quad b = d^{-1} a\, d,\quad d = b^{-1}\underline{a\,b\,a^{-1}}\,b \rangle$$

$$= \langle a, b \mid a = a\,b\,a^{-1}\,\underline{b^{-1} a\,b\,a^{-1} b}\,a\,b^{-1} a^{-1},\quad b = \underline{b^{-1} a\,b^{-1} a^{-1} b}\,a\,\underline{b^{-1} a\,b\,a^{-1} b} \rangle$$

$a^{-2}\cdot(\qquad)a$ $\qquad\qquad$ $b\,a^{-1}(\qquad)\cdot b^{-1} a\, b^{-1}$

$$= \langle a, b \mid 1 = a^{-1} b\, a^{-1} b^{-1} a\, b\, a^{-1} b\, a\, b^{-1},\quad 1 = b\, a^{-1} b^{-1} a\, b^{-1} a^{-1} b\, a\, b^{-1} a \rangle$$

inverses

$$= \boxed{\langle a, b \mid 1 = a^{-1} b\, a^{-1} b^{-1} a\, b\, a^{-1} b\, a\, b^{-1} \rangle}$$

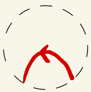
Theorem: Up to isomorphism, the fundamental group is an invariant of links

In other words, two diagrams coming from the same link give isomorphic groups.

__Proof__: We check invariance under the Reidemeister moves:

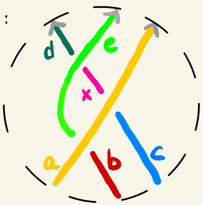
RI:

Relation: $a = a^{-1}aa$



Relation: ( )

RIII:



Relations: $\begin{cases} e = a^{-1}ba \\ x = a^{-1}ca \\ d = e^{-1}xe \end{cases}$

Eliminate x: $\begin{cases} e = a^{-1}ba \\ a^{-1}ca = ede^{-1} \end{cases}$

Eliminate e: $a^{-1}ca = a^{-1}bada^{-1}b^{-1}a$

$\Longleftrightarrow$

$\boxed{c = bada^{-1}b^{-1}}$

Relations: $\begin{cases} e = a^{-1}ba \\ y = b^{-1}cb \\ d = a^{-1}ya \end{cases}$

Eliminate y: $\begin{cases} e = a^{-1}ba \\ b^{-1}cb = ada^{-1} \end{cases}$

Eliminate e: $b^{-1}cb = ada^{-1}$

$\Longleftrightarrow$

$\boxed{c = bada^{-1}b^{-1}}$

__RII__: You will prove it in the exercises.

$\square$

Exercises: compute and explore fundamental groups