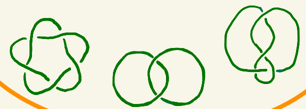
12. When two structures are secretly the same

We will be assigning

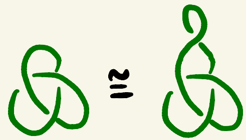
Links



$\rightsquigarrow$

Groups

C_n $\quad S_n$ $\quad \mathbb{Z}$ $\quad \mathrm{Sym}(\square)$

Just like we had a notion of equivalence of links: 

we have a notion of **isomorphism** of groups

For example, take $G = (\{1,-1\}, \cdot)$ and $H = (\{\text{True}, \text{False}\}, \text{XOR})$

$\cdot$	1	-1
1	1	-1
-1	-1	1

$\xleftrightarrow{\cong}$

intuitively the same

	T	F
T	T	F
F	F	T

How to make this mathematically precise?

Sets

Definition: an isomorphism of sets is a function $f : X \longrightarrow Y$ such that

1. Different elements get sent to different elements

   if $x \neq y$ then $f(x) \neq f(y)$

   Injectivity

2. Every element of Y is the image of some element of X.

   for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$

   Surjectivity

Example: $f : \{1, 2, 3\} \longrightarrow \{A, B, C\}$

$f(1) = A$

$f(2) = B$

$f(3) = C$

- Injectivity:
- Surjectivity:

We conclude that $\{1, 2, 3\} \cong \{A, B, C\}$ and we say they are isomorphic as sets

Important observation: take a finite set, for instance

$$X = \{ \text{members of this class} \}$$

Then we can find an integer $N \geq 0$ and an isomorphism

$$\{1, 2, 3, \ldots, N\} \longrightarrow X$$

<u>Definition</u>: the number N above is called the cardinality of X, and is denoted $|X|$.

In other words: every finite set is isomorphic (as sets) to one of:

$$\{\}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \ldots$$

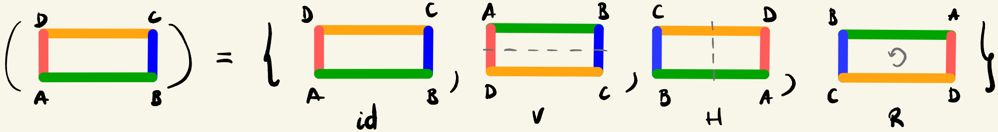
Thus we have reached a Classification of finite sets (high brow counting)

Groups

What does an isomorphism of groups look like?

Running example: Sym () and $C_2 \times C_2$

Recall that we had

Sym 

$$C_2 \times C_2 = \{(0,0), (1,0), (0,1), (1,1)\}$$

They have the same cardinality, so they are isomorphic as sets (good start)

How about the operations?

Operation tables:

Sym()

Sym( )	id	V	H	R
id	id	V	H	R
V	V	id	R	H
H	H	R	id	V
R	R	H	V	id

$C_2 \times C_2$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Sending

$\mathrm{id} \xmapsto{f} (0,0)$

$V \xmapsto{f} (1,0)$

$H \xmapsto{f} (0,1)$

$R \xmapsto{f} (1,1)$

the tables "agree", in other words:

$$\boxed{f(x * y) = f(x) * f(y)}$$

We arrive at:

Definition: Let G and H be groups. A function $f: G \to H$ is an isomorphism of groups if

1) f is an isomorphism of sets

2) For all $x, y \in G$, $f(x * y) = f(x) * f(y)$ "f respects the operation"

If such an isomorphism exists, we say that G and H are isomorphic and we write $G \cong H$.

A fundamental question in Group Theory is:

$\boxed{\text{Classify groups up to isomorphism}}$

This is a monumental task, unlike the case of finite sets

We can, however, take on more modest tasks.

Question: Classify all groups of cardinality 2

Step 1: We have $C_2 = (\{0, 1\}, +)$

Step 2: It seems that those are all the possibilities, so let's prove it.

Take some other group of cardinality 2. Say $G = \{a, b\}$.

Step 3: By the axioms, there is an identity element, let's say it's a. So we know: $a * a = a$

$a * b = b$

$b * a = b$

Step 4: What are the possibilities for $b * b$?

$b * b = a$ or $b * b = b$

$\Downarrow$ Then the map $0 \mapsto a$, $1 \mapsto b$ is an isomorphism

$\Downarrow$ Impossible:

Conclusion: there is exactly one group of cardinality 2 up to isomorphism

Exercises : classify up to isomorphism groups of small orders.