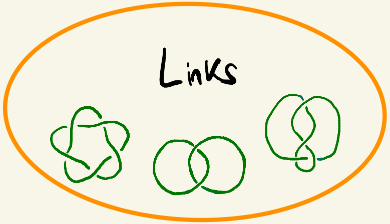
10. Surfaces attached to Knots.

Back to our goal:



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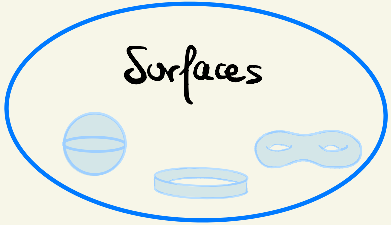
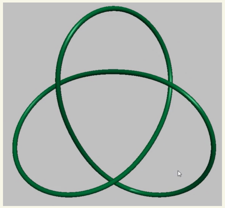


Idea (1930s): [Animation]

Question:



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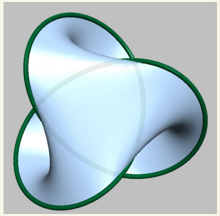
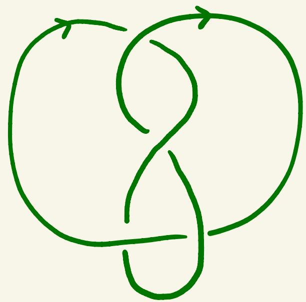


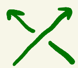
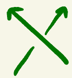
1934, Seifert:

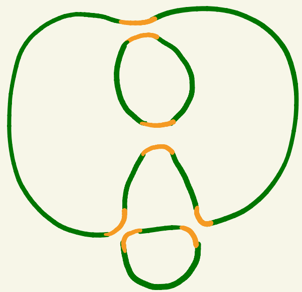
: "Here is an algorithm"

How does it work?

Step 1: Choose a diagram and orientation for the link:

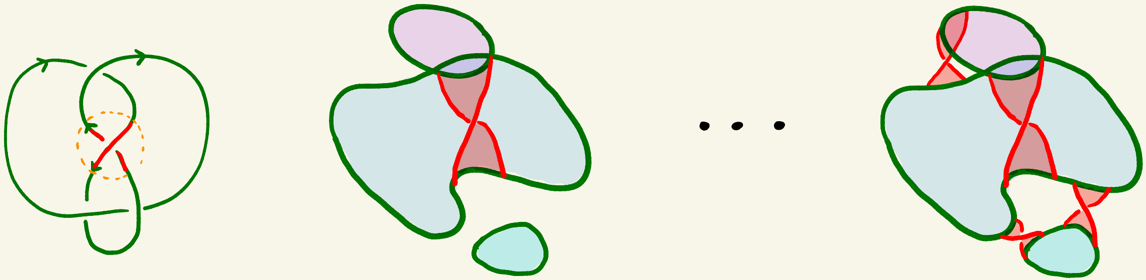


Step 2: smooth out every crossing:  and  become 

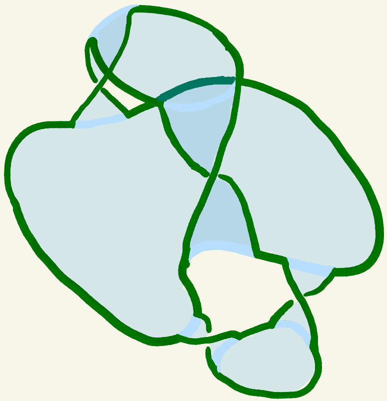


Step 3: make every resultant circle into a disk, and place the disks at different heights

Step 4: place twisted bands where there used to be crossings:



Result:



Remark: the result is always orientable.

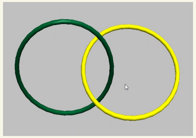
Definition: a surface whose boundary components form a link L is called a Seifert surface.

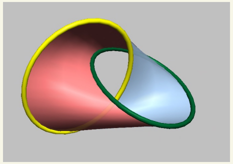
Example: the Seifert algorithm gives a Seifert surface

Warning: These are not unique!

The theory of surfaces gives us a new invariant:

genus





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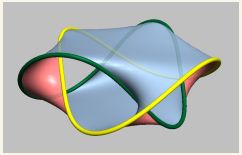

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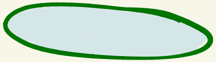
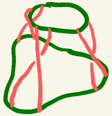
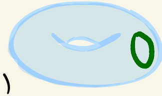
⋮

<u>Definition</u>: the genus of a link $g(L)$ is the minimum genus of a Seifert surface for L.

Examples:

- Unknot:  =  = $T_{0,1}$ $\quad$ 0 is obviously minimum.
- Trefoil:  $\rightsquigarrow$  = (Exercise)  = $T_{1,1}$

  so $g(\text{trefoil}) \leq 1$.

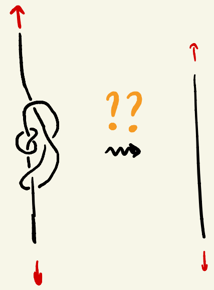
Observation: If $g(K) = 0$, then K = unknot. Proof: $g(K) = 0$ means that K is the boundary of $T_{0,1}$ = . But  is clearly unknotted.

Consequence: $g(\text{trefoil}) = 1$.

Note: Sage gives the genus of a link: `L.genus()`

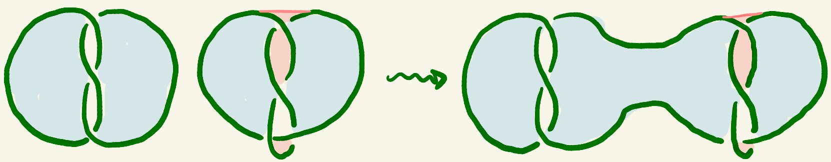
Recall our original question:

Can I untie a knot by knotting it more?



Crucial Theorem: $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Proof: Let S_1 and S_2 be Seifert surfaces of minimal genus. Note that $S_1 \# S_2$ is a Seifert surface for $K_1 \# K_2$:



We (you) proved that $g(S_1 \# S_2) = g(S_1) + g(S_2)$. Thus

$$g(K_1 \# K_2) \leq g(S_1 \# S_2) = g(S_1) + g(S_2) = g(K_1) + g(K_2).$$

It's a bit harder to prove that $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$, so we omit it.

$\square$

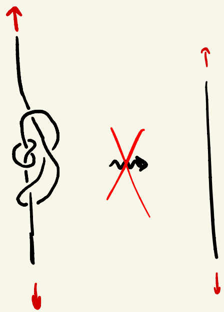
Corollary: If K_1 and K_2 are knots but $K_1, K_2 \neq$ unknot, then $K_1 \# K_2 \neq$ unknot.

Proof: $K_1 \neq \bigcirc \Rightarrow g(K_1) \geq 1$.

$K_2 \neq \bigcirc \Rightarrow g(K_2) \geq 1$.

Therefore $g(K_1 \# K_2) \underset{\text{Thm}}{=} g(K_1) + g(K_2) \geq 2 > 0 = g(\bigcirc)$

In other words:



Your second task:

- Investigate the genus of knots
- Prove the infinitude of prime knots.