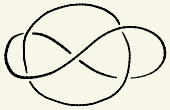
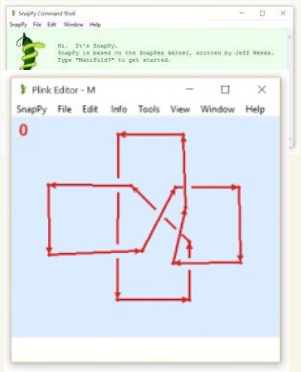
Reminder of knot theory so far:

- Knots, links and their diagrams



- Knot invariants: tricolorability, Jones polynomial.
- Computational knot theory:



Today: "conjectures fuel mathematical fields"

6. The value of conjectures

In any math field:

Exploration ⇝ Pattern ⇝ Conjecture ⇝ Proof (understanding)

$-t^4+t^3+t \leftrightarrow -\frac{1}{t^4}+\frac{1}{t^3}+\frac{1}{t}$

$t \leftrightarrow \frac{1}{t}$

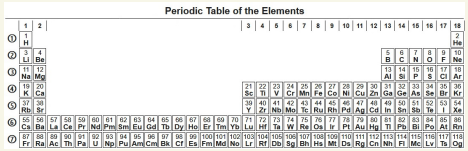
$\langle \times \rangle = A\langle) (\rangle + A^{-1}\langle \asymp \rangle$

$\langle \times' \rangle = A^{-1}\langle) (\rangle + A\langle \asymp \rangle$

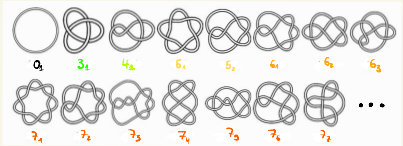
... ⇝ More exploration (more informed this time)

In knot theory history:

Peter Tait (1831-1901). Scottish physicist who thought maybe



⟷



(Spoiler: he was wrong)

Anyhow, he started the modern study of knots.

After much exploration, he posed three conjectures:

- Tait's first conjecture:

  Any reduced alternating link diagram has the smallest number of crossings

- Tait's second conjecture:

  Any two reduced reduced alternating connected diagrams for the same link have the same writhe.

- Tait's third conjecture:

  Any two reduced reduced alternating diagrams for the same link are related by a sequence of flypes.

No one knew how to prove these until the discovery of the Jones polynomial (1984)

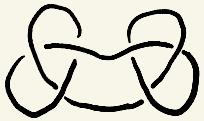
Proofs: 1987 (first and second), 1991 (third).

Alternating links:

Definition: A link diagram is alternating if following each strand's path, overcrossings and undercrossings alternate. A link is alternating if it has an alternating link diagram.

Example:

Nonexample:



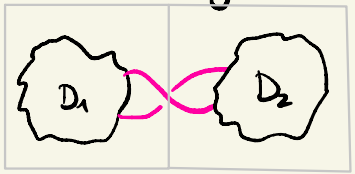
Fact: most prime knots are alternating:

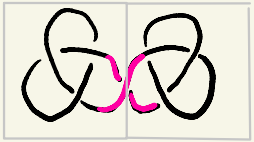
# crossings	# prime knots	# alternating prime knots
3	1	1
4	1	1
5	2	2
6	3	3
7	7	7
8	21	18
9	49	41
10	165	123

Q?

Reducibility, connectedness.

Definition: A link diagram D is reduced if we cannot find two arcs such that

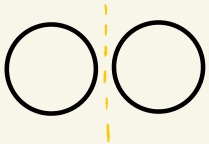
$D =$ 

Nonexample: 

Example:  . Notice if we had some twist  then we could untwist it, so then the trefoil could be represented with two crossings, which we can't do.

Definition: a link diagram is connected if ignoring (under/over) crossings, the diagram is connected.

Example: 

Nonexample: 

Q?

The first two conjectures, revisited:

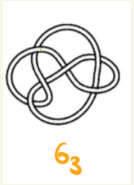
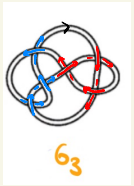
- **Tait's first conjecture:**

  Any reduced alternating link diagram has the smallest number of crossings

  Application:  cannot be drawn with ≤ 5 crossings.

- **Tait's second conjecture:**

  Any two reduced reduced alternating connected diagrams for the same link have the same writhe.

  Application:  $\neq$  . Proof:  writhe $= -2$,  writhe $= 0$

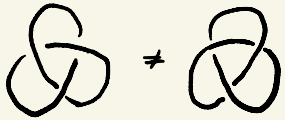
Q?

Amphichirality

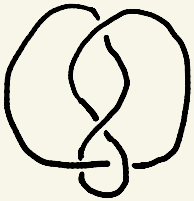
Definition: A link is **amphichiral** if it is equivalent to its mirror image.

In other words: changing all crossings $+1 \longleftrightarrow -1$ gives an equivalent link.

Nonexample:



Example:



https://www.mi.sanu.ac.rs/vismath/sl/l25.htm

Q? Exercises