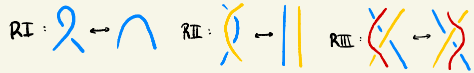
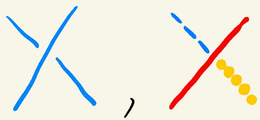
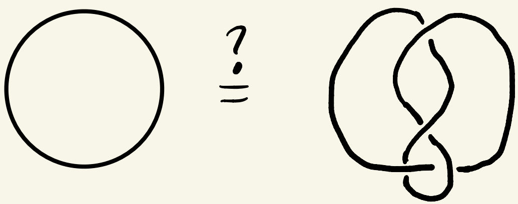
Reminder:

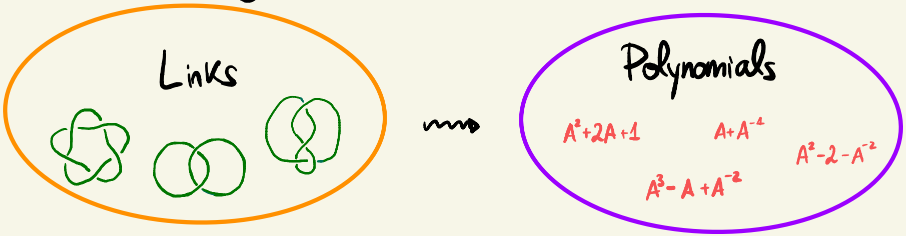
- Knots and links
- Showing they are equal: Reidemeister moves 

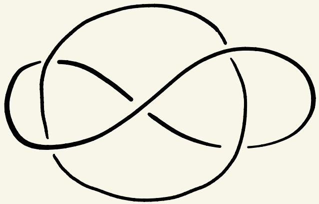
- Showing they are different: tricolorability 
- Problem: this is a weak invariant ("yes or no"), in that



4. A link polynomial

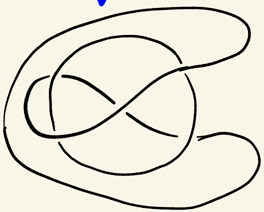
We will be assigning



$D =$ 

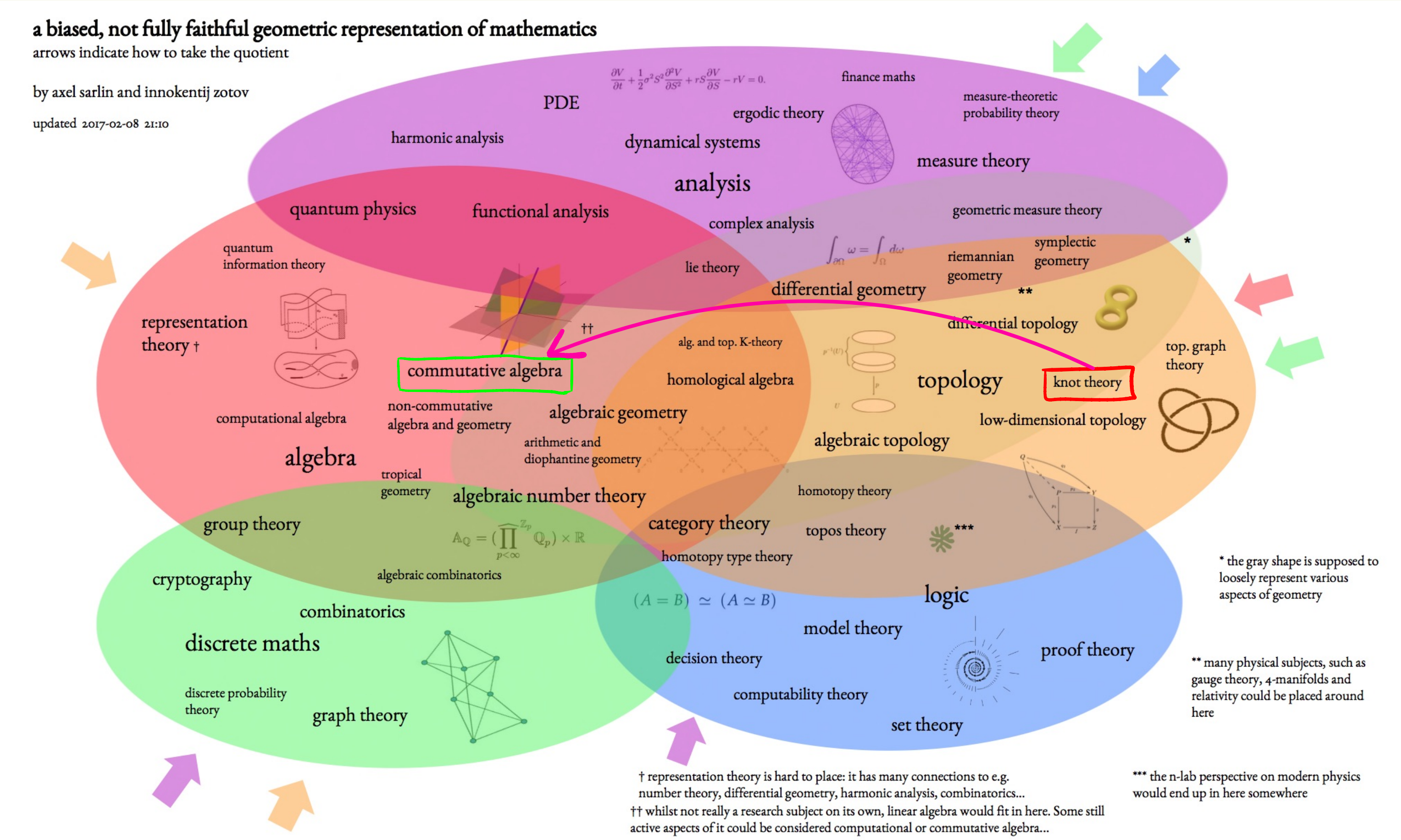
$$P_L(A) = A^{14} - 2A^{10} + 3A^6 - 2A^2 + A^{-2} - A^{-6}$$

↕ same link!

$D' =$  $\longmapsto$

$$P_{L'}(A) = A^{14} - 2A^{10} + 3A^6 - 2A^2 + A^{-2} - A^{-6}$$

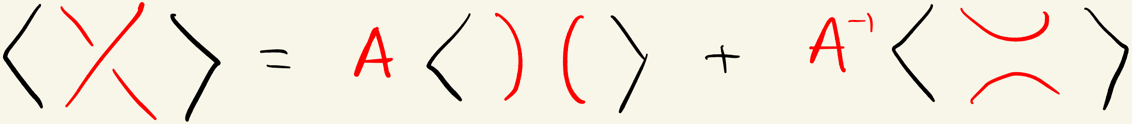
Q: polys?

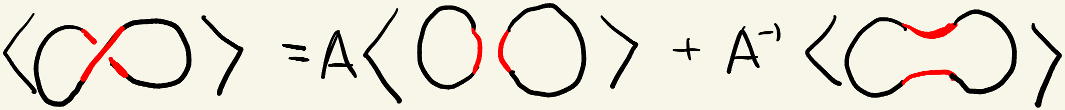

a biased, not fully faithful geometric representation of mathematics
arrows indicate how to take the quotient
by axel sarlin and innokentij zotov
updated 2017-02-08 21:10
$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$
finance maths
PDE
measure-theoretic
probability theory
ergodic theory
harmonic analysis
dynamical systems
measure theory
analysis
quantum physics
functional analysis
complex analysis
geometric measure theory
quantum
information theory
lie theory
$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$
riemannian
geometry
symplectic
geometry
*
differential geometry
**
representation
theory †
††
differential topology
alg. and top. K-theory
top. graph
theory
commutative algebra
homological algebra
topology
knot theory
computational algebra
non-commutative
algebra and geometry
algebraic geometry
low-dimensional topology
arithmetic and
diophantine geometry
algebraic topology
algebra
tropical
geometry
algebraic number theory
homotopy theory
group theory
category theory
topos theory
***
$\mathbb{A}_{\mathbb{Q}} = (\widehat{\prod_{p<\infty}}^{\mathbb{Z}_p} \mathbb{Q}_p) \times \mathbb{R}$
homotopy type theory
cryptography
algebraic combinatorics
logic
combinatorics
$(A = B) \simeq (A \simeq B)$
discrete maths
model theory
proof theory
decision theory
discrete probability
theory
graph theory
computability theory
set theory
* the gray shape is supposed to loosely represent various aspects of geometry
** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here
*** the n-lab perspective on modern physics would end up in here somewhere
† representation theory is hard to place: it has many connections to e.g. number theory, differential geometry, harmonic analysis, combinatorics...
†† whilst not really a research subject on its own, linear algebra would fit in here. Some still active aspects of it could be considered computational or commutative algebra...

How does it work?

Definition: The Kauffman bracket $\langle D \rangle$ of a link diagram D is given by the recipe:

(A is a formal variable)

- (crossing) 
- (unknot) $\langle \bigcirc \rangle = 1$
- (L+unknot) $\langle L \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle$

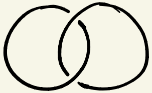
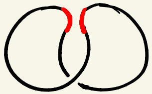
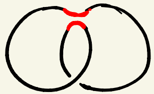
Example: 

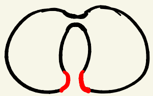
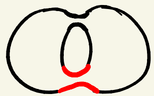
$$= A(-A^2 - A^{-2}) \langle \bigcirc \rangle + A^{-1} \cdot 1$$

$$= (-A^3 - A^{-1}) \cdot 1 + A^{-1}$$

$$= \boxed{-A^3}$$

Another example:

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= A \cdot (-A^3) + A^{-1} \cdot \big(A \langle$  $\rangle + A^{-1} \langle$  $\rangle \big)$

$= -A^4 + \langle$  $\rangle + A^{-2} \langle$  $\rangle$

$= -A^4 + 1 + A^{-2} \cdot (-A^2 - A^{-2}) \langle$  $\rangle$

$= -A^4 + 1 - 1 - A^{-4} = \boxed{-A^4 - A^{-4}}$

Q?

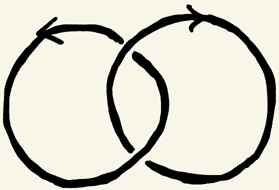
Kauffman bracket $\rightsquigarrow$ Jones polynomial

<u>Theorem</u> (Jones polynomial): Take an oriented link L with diagram D. (next page) Define

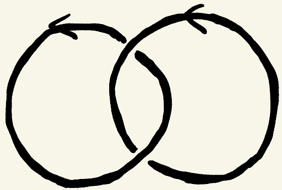
$$J(L) := (-A)^{-3\,\mathrm{writhe}(D)} \cdot \langle D \rangle$$

Then $J(L)$ is an invariant of links (does not depend on the diagram D)

Aside: take a link with orientations, such as:

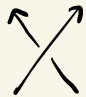


or



This is called an oriented link.

Crossings of L can be positive:

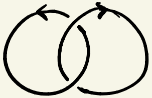


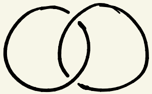
or

negative:

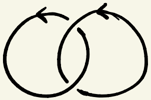
We say that the writhe of a diagram D for L is:

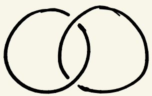
$$wr(D) = \#\text{positive crossings} - \#\text{negative crossings}$$

Example 1: $D =$  .

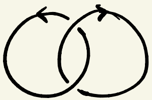
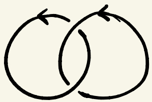
- $wr(D) = 2 - 0$
- $J(L) = (-A)^{-3\cdot 2} \cdot \langle$  $\rangle$

$$= A^{-6}\cdot(-A^{4} - A^{-4}) = \boxed{-A^{-2} - A^{-10}}$$

Example 2: $D =$  .

- $wr(D) = 0 - 2$
- $J(L) = (-A)^{-3\cdot(-2)} \cdot \langle$  $\rangle$

$$= A^{6}\cdot(-A^{4} - A^{-4}) = \boxed{-A^{10} - A^{2}}$$

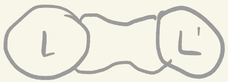
Note:  $\not\leftrightsquigarrow$  .

Q? Exercises

Remark: J is a much more powerful invariant than tricolorability (it takes more values): it actually distinguishes all prime knots with $c(K) \leq 9$

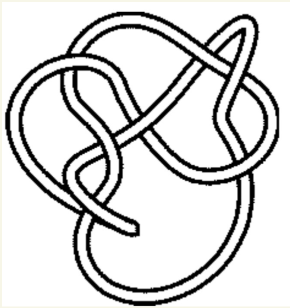
In the exercises, we will see many nice properties of J, such as:

$$J(L \# L') = J(L) \cdot J(L')$$

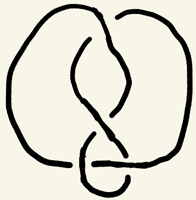
(Recall $L \# L' =$ 

)

Cautionary Tale



8_9



4_1

$$J(8_9) = (A^8 - A^4 + 1 - A^{-4} + A^{-8})^2$$

$$J(4_1) = A^8 - A^4 + 1 - A^{-4} + A^{-8}$$

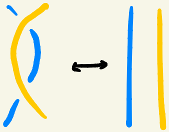
$$\Downarrow$$

$$J(4_1 \# 4_1) = J(8_9) \quad !$$

Why does it work?

Claim: J is an invariant of oriented links.

Proof: • RII invariance:



Note $\operatorname{writhe}(\cdots\ \cdots) = \operatorname{writhe}(\cdots\ \cdots) + 1 - 1$ ✓

So $J(\cdots\ \cdots) = -A^{-3\operatorname{writhe}(\cdots\ \cdots)} \cdot \langle\ \rangle$

Now $\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$

$= A\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right) + A^{-1}\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right)$

$= \langle\ \rangle + (A^2 + A^{-2} - (A^2+A^{-2}))\langle\ \rangle$

$= \langle\ \rangle$

Q?

Proof (ctd):

- RI invariance:

$$J(\ ) = (-A^{-3})^{\text{writhe}(\ )} \langle \ \rangle$$

$$= (-A^{-3})^{\text{writhe}(\curvearrowright)-1} \left(A \langle \ \rangle + A^{-1} \langle \ \rangle \right)$$

$$= (-A^{-3})^{\text{writhe}(\curvearrowright)} \cdot (-A^{3}) \cdot \left(A + A^{-1}(-A^{2} - A^{-2}) \right) \langle \cap \rangle$$

$$= (-A^{-3})^{\text{writhe}(\curvearrowright)} \cdot (-A^{3}) \cdot (-A^{-3}) \langle \cap \rangle$$

$$= (-A^{-3})^{\text{writhe}(\curvearrowright)} \cdot \langle \cap \rangle$$

$$= J(\curvearrowright)$$

- RIII invariance: in the exercises.

Q?

But really, how would you come up with this?