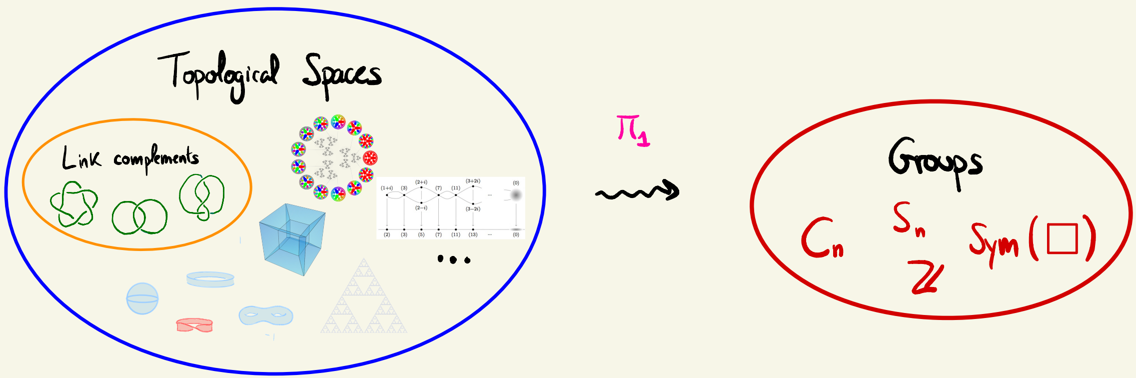
14. The fundamental group of a space

There is a more general notion of π, which extends to:

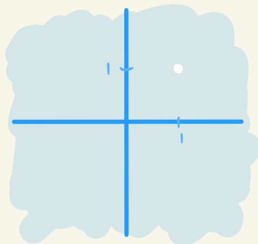


Topological spaces are the most general notion where "continuity" is defined.

Idea: take a space X and fix a point $x_0 \in X$. Then

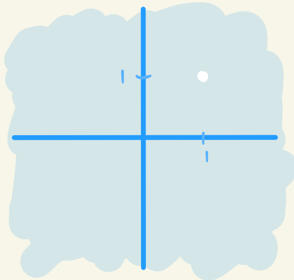
$$\pi_1(X) = \{\text{ paths on } X \text{ starting and ending at } x_0,\ \text{up to homotopy} \}$$

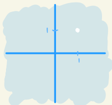
To illustrate this, take $X = \mathbb{R}^2 \setminus \{(1,1)\}$:



Path, homotopy: [Desmos]

Observation: we can compose paths!



In fact, $\pi_1\Big($  $\Big)$ is a group!

Associativity easy, identity element + inverses: in the exercises.

Fact: let X be any topological space. Then $\pi_1(X)$ is a group, called the

Fundamental group of X

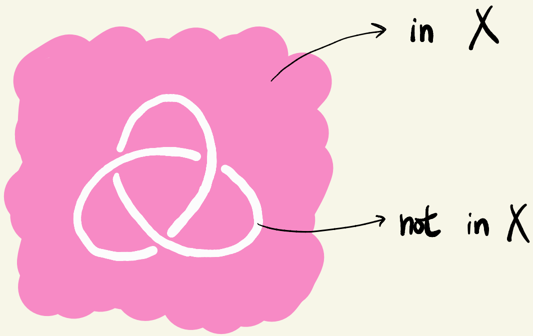
(the beginning of algebraic topology)

So today we defined

- $\pi(K)$ for a link K
- $\pi_1(X)$ for a space X

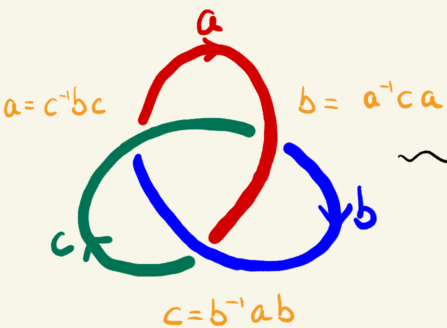
the same?

Almost: given K, consider $\mathbb{R}^3 \setminus K$:



Then $\pi(K) = \pi_1(\mathbb{R}^3 \setminus K)$

But our definition didn't mention paths...

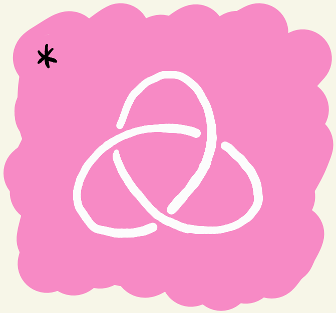


$\rightsquigarrow \pi(\text{trefoil}) = \langle a, b, c \mid a = c^{-1}bc,\ b = a^{-1}ca,\ c = b^{-1}ab \rangle$

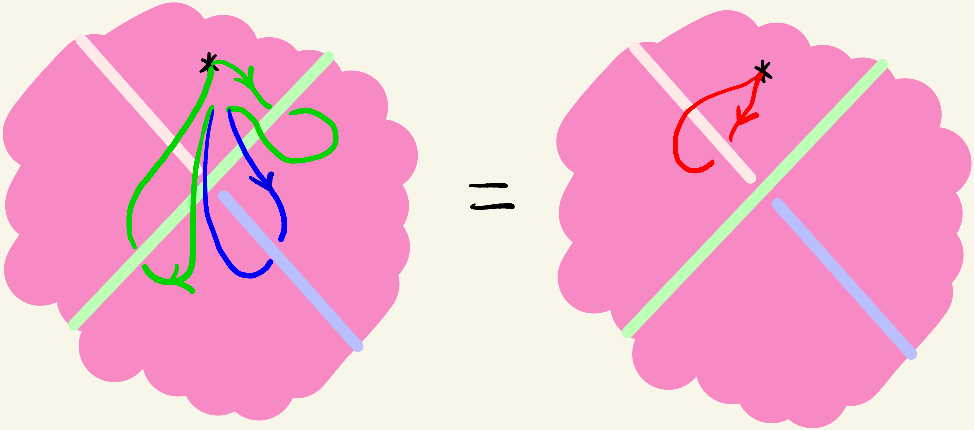
Our weird definition came from:

Generators:

one per arc



Relations:



In the exercises: explore fundamental groups of spaces

(not knots!)