Proposition: The categorification of the virtual braid group (Thiel 2009) can be emulated for all virtual Artin groups (as defined by Bellingeri-Paris-Thiel, 2021)

Proof: the only new relation which is nontrivial to check is:

(v3) $\mathrm{Prod}_R(\tau_s, \tau_t, m_{s,t}-1)\,\sigma_s = \sigma_r\,\mathrm{Prod}_R(\tau_s, \tau_t, m_{s,t}-1)$, where $r = s$ if $m_{s,t}$ is even and $r = t$ if $m_{s,t}$ is odd, for $s, t \in S$, $s \neq t$ and $m_{s,t} \neq \infty$.

m odd : $\underbrace{\tau_s \tau_t \cdots \tau_s \tau_t}_{m-1} \sigma_s = \sigma_t \underbrace{\tau_s \tau_t \cdots \tau_s \tau_t}_{m-1}$

$$F(\tau_s \tau_t \cdots \tau_t \sigma_s) = R_{st\cdots t}(2) \longrightarrow \underline{R_{st\cdots t} \otimes_R B_s} = R_{st\cdots t} \otimes_{R^s} R$$

$$1 \longmapsto 1 \otimes \alpha_s + st\cdots t(\alpha_s) \otimes 1$$

$$F(\sigma_t \tau_s \tau_t \cdots \tau_t) = R_{st\cdots t}(2) \longrightarrow \underline{B_t \otimes_R R_{st\cdots t}} = R \otimes_{R^t} R_{st\cdots t}$$

$$1 \longmapsto 1 \otimes \alpha_t + \alpha_t \otimes 1$$

Lemma 3.2: (in Thiel 2009)

$$\psi : R_{st\cdots t} \otimes_R B_s \xrightarrow[\cong]{a \otimes b \mapsto a \otimes st\cdots t(b)} R \otimes_R R_{\underbrace{st\cdots tsts\cdots s}_{2m-1}} R_{sts\cdots s} = R \otimes_{R^t} R_{sts\cdots s}$$

Claim: we have an isomorphism of complexes given by:

$$\begin{array}{ccc} R_{st\cdots t}(2) & \xrightarrow{d} & R_{st\cdots t} \otimes_{R^s} R \\ \big\downarrow \mathrm{id} & & \big\downarrow \psi \\ R_{sts\cdots t}(2) & \xrightarrow{d'} & R \otimes_{R^t} R_{st\cdots t} \end{array}$$

$$1 \xmapsto{d} \tfrac{1}{2}(1 \otimes \alpha_s + st\cdots t\alpha_s \otimes 1)$$

$$\xmapsto{\psi} \tfrac{1}{2}(1 \otimes st\cdots t\,\alpha_s + st\cdots t\alpha_s \otimes 1)$$

$$1 \xmapsto{d'} \tfrac{1}{2}(1 \otimes \alpha_t + \alpha_t \otimes 1)$$

equal since $\underbrace{st\cdots st}_{m-1}\,\alpha_s = \alpha_t$

m even: $\underbrace{T_s T_t \cdots T_t T_s}_{m-1} \sigma_s = \sigma_s \underbrace{T_s T_t \cdots T_t T_s}_{m-1}$

$$F(T_s T_t \cdots T_s \sigma_s) = R_{st\cdots s}(2) \longrightarrow \underline{R_{st\cdots s} \otimes_R B_s}$$

$$1 \longmapsto 1 \otimes \alpha_s + st\cdots s(\alpha_s) \otimes 1$$

$$F(\sigma_s T_s T_t \cdots T_s) = R_{st\cdots s}(2) \longrightarrow \underline{B_s \otimes_R R_{st\cdots s}}$$

$$1 \longmapsto 1 \otimes \alpha_s + \alpha_s \otimes 1$$

Lemma 3.2: $\psi : R_{st\cdots s} \otimes_R B_s \xrightarrow[\cong]{a\otimes b \mapsto a \otimes st\cdots t(b)} R \otimes_{R^{st\cdots ts s sts\cdots s}} R_{sts\cdots s} = R \otimes_{R^{tst}} R_{sts\cdots s}$

↳ Problem: we want the map to land in $R \otimes_{R^s} R_{sts\cdots s}$

New lemma: $\varphi_0 : R \otimes_{R^{tst}} R \xrightarrow{\cong} R \otimes_{R^s} R$

$$a \otimes_{R^{tst}} b \longmapsto t(a) \otimes_{R^s} t(b)$$

if $tstc = c$ then $st(c) = t(c)$

Well defined: $ac \otimes_{R^{tst}} b \longmapsto t(a)t(c) \otimes_{R^s} t(b) = t(a) \otimes_{R^s} t(c)t(b)$

$a \otimes_{R^{tst}} cb \longmapsto t(a) \otimes_{R^s} t(c)t(b)$

Let $\varphi = (\varphi_0) \otimes R_{st\cdots s}$ PS: φ_0 is not a bimodule homomorphism. I'm still unsure about this case.

Claim: we have an isomorphism of complexes given by:

$$\begin{array}{ccc} R_{st\cdots s}(2) & \xrightarrow{d} & R_{st\cdots s} \otimes_{R^s} R \\ \big\downarrow -\mathrm{id} & & \big\downarrow \varphi \circ \psi \\ R_{sts\cdots s}(2) & \xrightarrow{d'} & R \otimes_{R^t} R_{st\cdots s} \end{array}$$

$$1 \overset{d}{\longmapsto} \tfrac{1}{2}(1 \otimes \alpha_s + st\cdots s\alpha_s \otimes 1)$$

$$\overset{\psi}{\longmapsto} \tfrac{1}{2}(1 \otimes st\cdots s\alpha_s + st\cdots s\alpha_s \otimes 1)$$

$$\overset{\varphi}{\longmapsto} \tfrac{1}{2}(1 \otimes ts\cdots ts\alpha_s + tst\cdots ts\alpha_s \otimes 1)$$

$$1 \overset{d'}{\longmapsto} -\tfrac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1)$$

equal since $\underbrace{ts\cdots ts}_{m}\alpha_s = -\alpha_s$ □