Guided exercise: $\widetilde{SL_2}(\mathbb{R})$ is not a matrix group.

Let $\Phi : \widetilde{SL_2}(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ be a Lie group homomorphism. You will prove that $\operatorname{Ker} \Phi \neq 1$.

(a) Define a functor $\mathbb{C} \underset{\mathbb{R}}{\otimes} - :$ Lie algebras$_{\mathbb{R}} \longrightarrow$ Lie algebras$_{\mathbb{C}}$ such that

$$\begin{tikzcd} \text{Lie alg}_{\mathbb{R}} \arrow[r, "\mathbb{C}\otimes -"] \arrow[d, "\text{forget}_{\mathbb{R}}"] & \text{Lie alg}_{\mathbb{C}} \arrow[d, "\text{forget}_{\mathbb{C}}"] \\ \text{Vect}_{\mathbb{R}} \arrow[r, "\mathbb{C}\otimes -"] & \text{Vect}_{\mathbb{C}} \end{tikzcd}$$

commutes.

This means that we have a natural isomorphism of functors $(\mathbb{C}\otimes -) \circ \text{forget}_{\mathbb{C}} = \text{forget}_{\mathbb{R}} \circ (\mathbb{C} \otimes -)$.

This is called complexification.

(b) Take $\varphi = \operatorname{Lie}(\Phi)$ and complexify it to get $\varphi_{\mathbb{C}} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$. Argue that you can lift this to a real Lie group homomorphism $\psi : SL_2(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$.

(c) Restrict ψ to $SL_2(\mathbb{R}) \subseteq SL_2(\mathbb{C})$ to get $\psi_{\mathbb{R}} : SL_2(\mathbb{R}) \to GL_n(\mathbb{C})$. Prove that $\operatorname{Lie}(\psi_{\mathbb{R}}) = \varphi$.

(d) Let $\pi : \widetilde{SL_2}(\mathbb{R}) \longrightarrow SL_2(\mathbb{R})$ be the projection map. Use the fully-faithfulness of $\operatorname{Lie}(-)$ to show that $\psi_{\mathbb{R}} \circ \pi = \Phi$.

(e) Conclude that $\operatorname{Ker}(\Phi) \supseteq \operatorname{Ker}(\pi)$ and therefore Φ is not injective.

Remark: complexifications are also defined for Lie groups, and in this case the complexification of $\widetilde{SL_2}(\mathbb{R})$ is actually $SL_2(\mathbb{C})$.