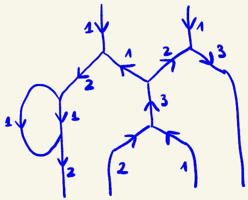
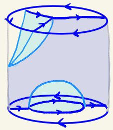
The deal with foams II

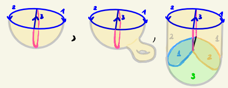
0. Reminder of last time:

- Webs  $\longleftrightarrow$ $\mathfrak{gl}_n$- intertwiners between $\otimes$ of $\Lambda^i V$

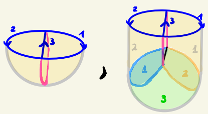
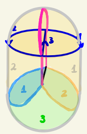
Evaluation of closed webs: $\Gamma \mapsto \langle \Gamma \rangle \in \mathbb{Z}[q,q^{-1}]$

- Foams : cobordisms between webs 

$\mathbb{Z}[x_1,\dots,x_n]$ -mod with basis

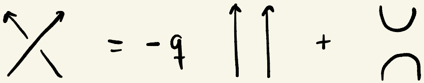
$\{$  $, \dots \}$

- Universal construction : web $\Gamma \rightsquigarrow \mathcal{F}_n(\Gamma) = W_n / \mathrm{Ker}((\,,)_n)$

where $\Big($  $\Big)_n := \Big\langle$  $\Big\rangle_n \in \mathbb{Z}[x_1,\dots,x_n]^{S_n}$ Foam evaluation

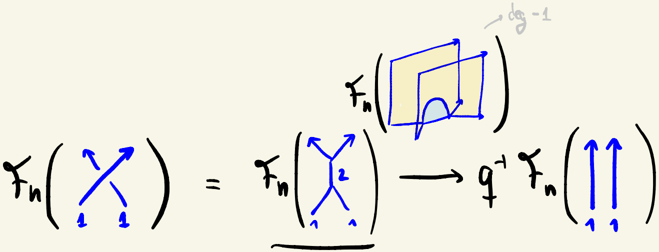
- $\mathcal{F}_n(\Gamma)$ categorifies $\langle \Gamma \rangle$, i.e. $q\dim(\mathcal{F}_n(\Gamma)) = \langle \Gamma \rangle$.

- Braiding identity $\rightsquigarrow$ $\mathfrak{gl}_n$-polynomial of links $P_n(q)$



$P_n(q) = \mathrm{HOMFLY}(q, t = q^n)$

- Rickard complexes $\rightsquigarrow$ Complex $C_*(L)$ with $\chi_q(C_*(L)) = P_n(q)$



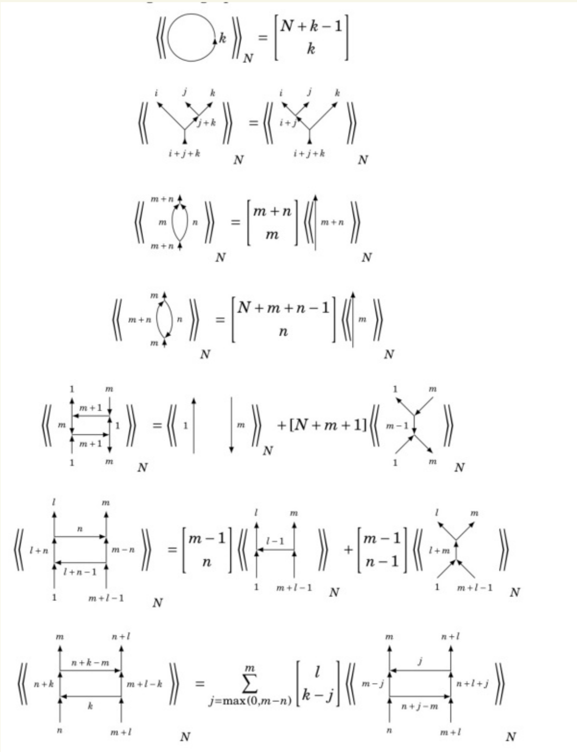
$C_*(L) = \bigotimes_{\text{crossings}}$ Rickard complex

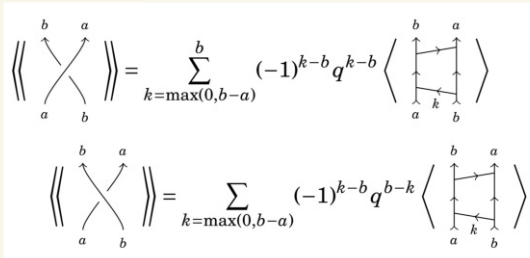
(This agrees with KhR for $\mathfrak{sl}_N$)

Today:
- Describe Soergel bimodules (type A) in terms of foams, as well as HH_0.
- Describe a "symmetric" Khovanov-Rozansky link homology for $\mathrm{HOMFLY}(q^N, q)$

1. Symmetric MOY calculus

Analogous to (exterior) MOY calculus for $\Lambda^i V$ but with $S^i V$ for sl_N:





Some remarks:

- In the uncolored case this gives the same polynomial as the exterior calculus

  However, $\left\langle\!\left\langle \circlearrowleft_{S^aV} \right\rangle\!\right\rangle_N = \begin{bmatrix} N+a-1 \\ a \end{bmatrix}$ whereas $\left\langle \circlearrowleft_{\Lambda^aV} \right\rangle_N = \begin{bmatrix} N \\ a \end{bmatrix}$

- Theorem (Wu, 2014): $\langle\!\langle$ closed web $\rangle\!\rangle_N$ = Value of closed web as an intertwiner $\mathbb{C}[q^{\pm 1}] \to \mathbb{C}[q^{\pm 1}]$

- (As far as I Know): It's open to show that the functor from the spider category:

$$Sp_{sym}(N) = \begin{cases} \text{Objects: sequences of integers} \\ \text{Morphisms: webs between them modulo symmetric MOY relations} \end{cases}$$

(Expected in some form)

is an equivalence.

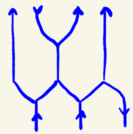
If so, then $Kar(Sp_{sym}(N)) \cong U_q(sl_N)$-mod.

2. Two special kinds of webs

Definition: Let $\underline{k}_0$ and $\underline{k}_1$ be sequences of integers.

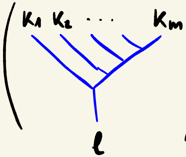
A $\underline{k}_0$-web-$\underline{k}_1$ is a web Γ between them:

$\underline{k}_0 = (1,-1,1,3) \rightarrow$ level $1-1+1+3=4$



$\underline{k}_1 = (2,3,-1) \rightarrow$ level $2+3-1=4$

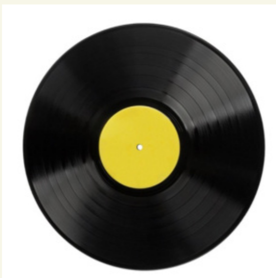
A $\underline{k}_0$-web-$\underline{k}_1$ is braid-like if all arrows point up

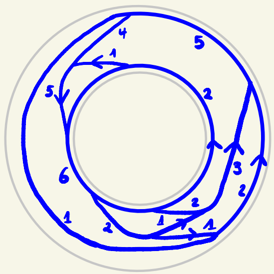
Fun fact: $\mathrm{Hom}_{\mathbf{Sp}_{\mathrm{sym}}(N)}(\ell, \overset{\text{level } \ell}{\underline{k}}) = \mathbb{Z}[q,q^{-1}]\cdot \Bigg($ 

$\Bigg)$

"Multiplicity of $S^\ell V$ in $S^{k_1}V \otimes \dots \otimes S^{k_m}V$ is 1"

Definition: A vinyl graph is a web in the annulus whose arrows rotate "positively":

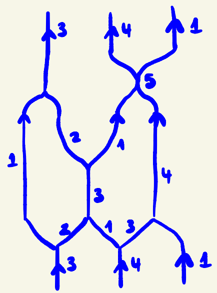


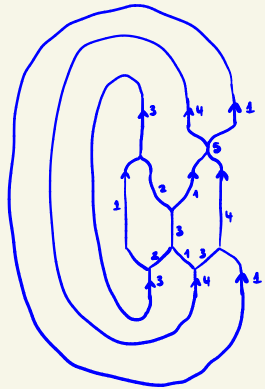


level = "# tracks"

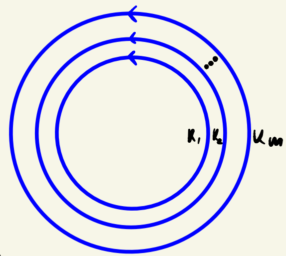
= rot(web) = signed # circles

Remark: clearly every vinyl graph is the closure of a braid-like web:

$\Gamma =$ 

$\hat{\Gamma} =$ 

Theorem (Queffelec-Rose, 2016): The $\mathbb{Z}[q^{\pm 1}]$-module generated by vinyl graphs of level ℓ is generated by:

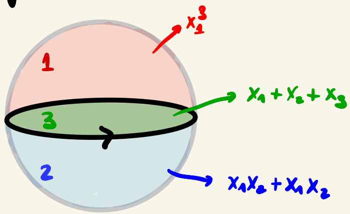


for all tuples with $\ell = k_1 + \dots + k_m$.

3. Two special kinds of foams

Disk-like foams (aka HOMFLY-PT foams)

Recall that foams are CW-complexes with thickness



Decorations are symmetric polynomials in #variables = thickness

And there is a monoidal functor $\mathcal{F}_N : \mathrm{Foam}_N \longrightarrow \underbrace{\mathbb{Z}[x_1,\dots,x_N]}_{\mathbb{Z}[\mathbb{X}]}\text{-gmod}$

where Foam_N : • Objects: webs

• Morphisms: foams

Equivalently: 2-functor "Foam_N" $\rightarrow$ "$\mathbb{Z}[\mathbb{X}]$-mod"

"Foam_N":

• Objects: finite sequences $\underline{k}$

• Morphisms: webs

• 2-morphisms: foams

$\circ_0 : \otimes \qquad \circ_1 : \circ$

"$\mathbb{Z}[\mathbb{X}]$-mod":

• Object: $*$

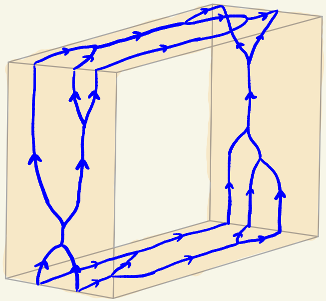
• Morphisms: Graded $\mathbb{Z}[\mathbb{X}]$-modules

• 2-morphisms: graded homomorphisms

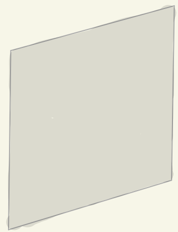
$\circ_0 : \otimes \qquad \circ_1 : \circ$

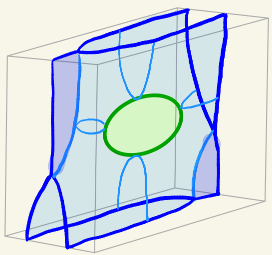
Definition: A *disk-like foam* F is a foam in a fixed $[0,1]^3$ such that:

- The boundary lies on the shaded region, and it consists of braid-like webs:

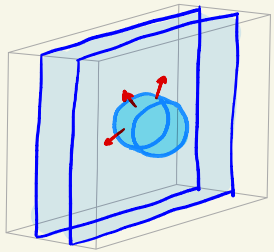


- Normal vectors to the foam are never parallel to the plane



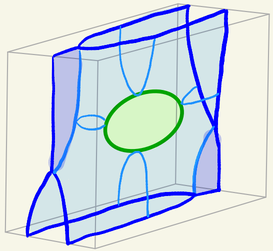


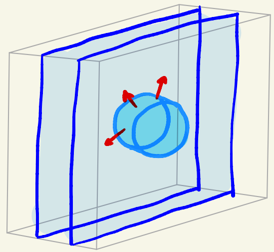
disk-like



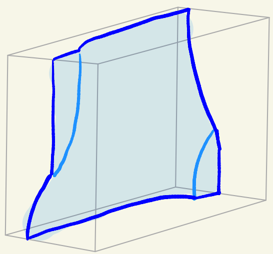
not disk-like

Remark: they are called disk-like because each "sheet" consists of a disk touching all four sides:

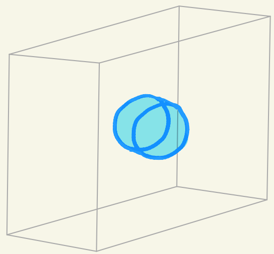




e.g:

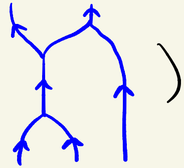


Free floating sphere, not a disk:

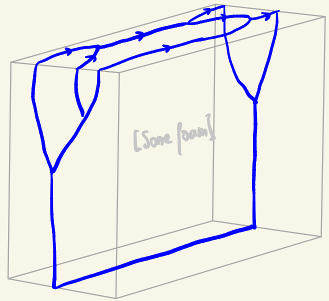


Also, no holes, etc.

Remark:

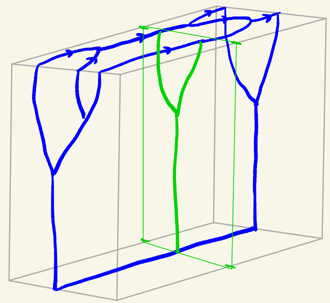
A special kind of disk-like foam: fix a braid-like web Γ (e.g. )

Then:
- Γ is on the top
- Standard trees are on the sides
- A single strand on the bottom



"**Rooted Γ-foam**"

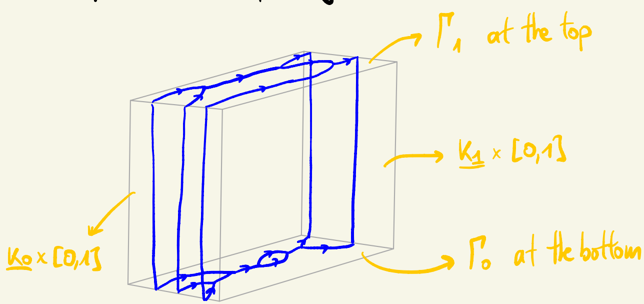
It is called **tree-like** if each slice is a tree:



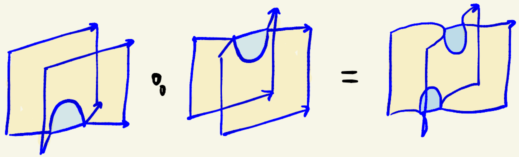
Fact: a rooted Γ-foam is ∞-equivalent to a $\mathbb{Z}$-linear combination of tree-like ones.

Define a 2-category DLF_N

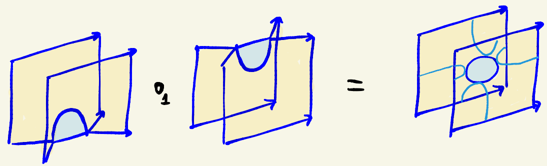
- Objects: positive integer sequences $\underline{k}$ of level N
- Morphisms: Braid-like webs between them $\Gamma : \underline{k_0} \to \underline{k_1}$ (level N)
- 2-morphisms: only between pairs of $\underline{k_1}$-webs-$\underline{k_0}$ Γ_0, Γ_1 so that:



Horizontal composition:



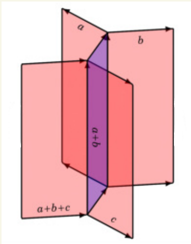
Vertical composition:



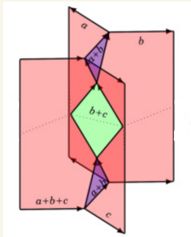
Recall the functor $\mathcal{F}_N : \mathbf{Foam} \longrightarrow \mathbb{Z}[x_1, \dots, x_N]^{S_N}\text{-gmod}$.

If two foams $F, G : \emptyset \to \Gamma$ have $\mathcal{F}_N(F) = \mathcal{F}_N(G)$, they are N-equivalent.

Some foams are N-equivalent for all N, e.g. we can replace



by



and $\mathcal{F}_N$ won't see it.

"∞-equivalent"

Definition: Form the 2-category $\widehat{\mathbf{DLF}}_N$ by extending $\mathbb{Q}$-linearly the 2-morphisms and modding out by ∞-equivalence.

Remark: 2-hom spaces are graded: $\deg(F) = \deg^{\wedge}(F) - \frac{\|K_0\|^2 + \|K_1\|^2 + 2N^2}{4}$

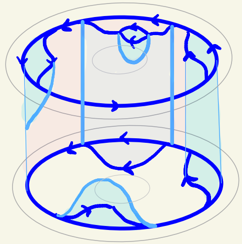
Horizontal and vertical compositions are additive in this grading.

Vinyl foams (aka symmetric foams)

Braid-like webs $\longrightarrow$ Disk-like foams

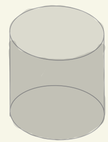
Vinyl graphs $\longrightarrow$ Vinyl foams

Example:



Remarks:

- Also require normal vectors to not be 'parallel to the cylinder'



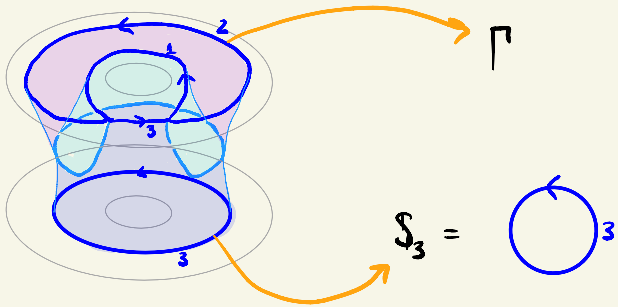
  ↳ Each "sheet" consists of a cylinder, called a tube.

- Cutting up a vinyl foam along a radius gives a disk-like foam.

- The category TLF_k :
  - Objects: vinyl graphs of level k
  - Morphisms: vinyl foams

Remark:

The analog of the rooted Γ-foam is the vinyl Γ-foam- $\mathbb{S}_k$:



It's also called tree-like if the slices are trees.

4. Singular Soergel bimodules via foams

Let $R = \mathbb{Q}[x_1, \dots, x_N]$, and for $T \subseteq S_n$, denote $R^T = \{T\text{-invariant polynomials in } R\}$
deg 2

Write grading shifts as $M(n) = Mq^n$. Let $S = \{s_i = (i\ i{+}1)\} \subseteq S_n$.

Define the 2-category $\mathrm{SBSBim}_N \subset R\text{-bim}$

- Objects: subsets of $\{1, \dots, n-1\} \longleftrightarrow$ subsets $I \subseteq S$ (parabolic subgroups of S_n) $\longleftrightarrow \{R^I$ as an (R^I, R^I)-bimodule$\}$
- Morphisms: Induction and restriction functors, i.e. tensors of $\underbrace{{}_{R^J}R^J{}_{R^I} \otimes (-)\, q^{s(I,J)}}_{\text{Ind}}$ or $\underbrace{{}_{R^I}R^I{}_{R^J} \otimes (-)\, q^{s(I,J)}}_{\text{Res}}$
- 2-morphisms: bimodule maps

where $J = I \sqcup \{s\}$

Examples of 1-morphisms for $N=3$:

- $\emptyset \longrightarrow \{s_1\} \longrightarrow \emptyset \longrightarrow \{s_2\} \longrightarrow \emptyset$

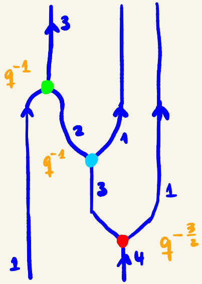
$$R^{\emptyset} \otimes_{R^{s_1}} R^{s_1} \otimes_{R^{s_1}} R^{\emptyset} \otimes_{R^{s_2}} R_{s_2} \otimes_{R^{s_2}} R^{\emptyset}\, q^{-2} \quad (= BS(s_1 s_2))$$

- $\{s_1\} \longrightarrow \{s_1, s_2\} \longrightarrow \{s_2\} \longrightarrow \emptyset$

$$R^{\emptyset} \otimes_{R^{s_2}} R^{s_2} \otimes_{R^{s_1,s_2}} R^{s_1,s_2} \otimes_{R^{s_1,s_2}} R^{s_1}\, q^{-\frac{5}{2}}$$

Another way to depict singular Bott-Samelson bimodules:

Γ braid-like $\underline{k}_1$-web-$\underline{k}_0$ $\rightsquigarrow$ $\mathcal{B}(\Gamma) \in \mathrm{Hom}_{\mathrm{SBSBim}}(\underline{k}_0, \underline{k}_1)$

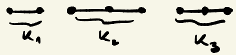


$${}_{R^{s_1,s_2}}R^{s_2} \otimes_{R^{s_2}} R^{s_2} \otimes_{R^{s_2,s_3}} R^{s_2,s_3}{}_{R_{s_2,s_3,s_4}} \; q^{-\frac{7}{2}}$$

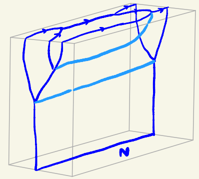
Finally, define $\mathbb{S}\mathrm{SBim} := (\oplus, \overset{\subseteq}{\oplus}, q^{\frac{a}{2}})$-completion of SBSBim.

Remark: $\mathrm{Hom}_{\mathrm{SBSBim}}(\emptyset, \emptyset) = \mathrm{BSBim}$ and $\mathrm{Hom}_{\mathbb{S}\mathrm{SBim}}(\emptyset, \emptyset) = \mathrm{SBim}$.

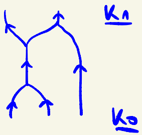
We construct a 2-functor $\mathcal{F}^D_\infty : \mathrm{DLF}_N \longrightarrow \mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$

$\underline{k}$ (positive, level N) $\longmapsto \mathcal{F}^D_\infty(\underline{k}) =$ 

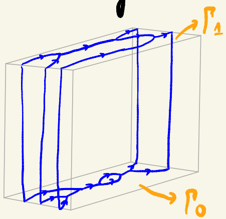
(Generators of parabolic subgroup)

Γ braid-like $\longmapsto \mathcal{F}^D_\infty(\Gamma) = R^{S_N} \cdot \{$ rooted Γ-foams 

$\} \,/\, \infty$-equivalence

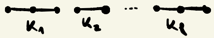


F disk-like foam $\longmapsto \mathcal{F}^D_\infty(F) : \mathcal{F}^D_\infty(\Gamma_0) \longrightarrow \mathcal{F}^D_\infty(\Gamma_1)$ given by $F \circ (-)$

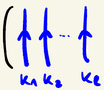


A priori, it's unclear that this maps to $\mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$

Proposition (Robert–Wagner, 2019) The following hold:

1. Let $\underline{k}$ be positive of level N. Let I be 

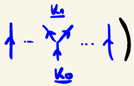
(generators of parabolic subgroup)

   $\mathcal{F}^D_\infty\Big($ 

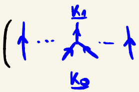
$\Big)$ has an R^{S_n}-algebra structure and is isomorphic to R^I as an algebra.

2. Let $\underline{k_0}, \underline{k_1}$ be positive of level N and let $I_0, I_1 \subseteq S_n$ the corresponding sets of generators.

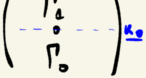
   $\mathcal{F}^D_\infty(\Gamma)$ has a (R^{I_1}, R^{I_0})-bimodule structure

3. $\mathcal{F}^D_\infty\Big($ 

$\Big) \cong {}_{R^{I_1}}R^{I_0}{}_{R^{I_0}}$ as bimodules

   $\mathcal{F}^D_\infty\Big($ 

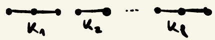
$\Big) \cong {}_{R^{I_0}}R^{I_1}{}_{R^{I_1}}$ as bimodules

4. $\mathcal{F}^D_\infty\Bigg($ 

$\Bigg) \cong \mathcal{F}^D_\infty(\Gamma_1) \otimes_{R^{I_0}} \mathcal{F}^D_\infty(\Gamma_0)$

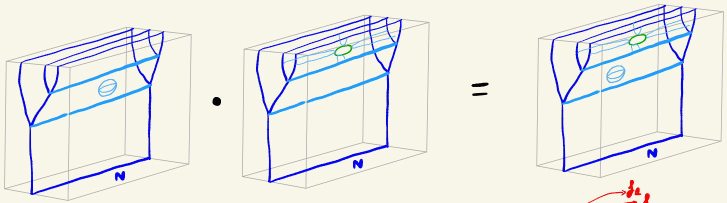
Corollary: The 2-functor $\mathcal{F}^D_\infty$ is well-defined

Sketch of proof:

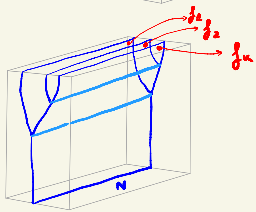
1. Let $\underline{k}$ be positive of level N. Let I be  (generators of parabolic subgroup)

$\mathcal{F}^D_\infty\left(\uparrow_{k_1}\uparrow_{k_2}\cdots\uparrow_{k_l}\right)$ has an R^{S_n} - algebra structure and is isomorphic to R^I as an algebra.

Idea:

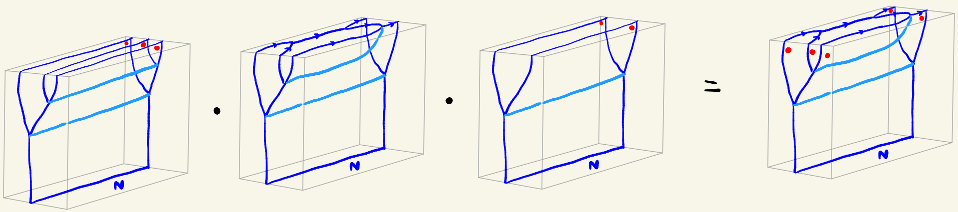


Isomorphism is given by $f_1 \otimes \dots \otimes f_k \longmapsto$

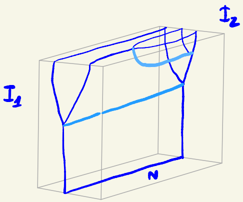


2. Let $\underline{k_0}$, $\underline{k_1}$ be positive of level N and let $I_0, I_1 \subseteq S_n$ the corresponding sets of generators.

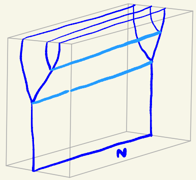
$\mathcal{F}^D_\infty(\Gamma)$ has a (R^{I_1}, R^{I_0}) - bimodule structure:



3. $\mathcal{F}_\infty^D\Big($  $\Big) \cong {}_{R^{I_1}}R^{I_0}{}_{R^{I_0}}$ as bimodules



$\rightsquigarrow$ bend



$\overset{\mathcal{F}_\infty^D}{\rightsquigarrow} \; {}_{R^{I_1}}R^{I_0}{}_{R^{I_2}}$

Similarly for 

$\square$

Fact: $\mathcal{F}_\infty^D : \mathrm{DLF}_N \longrightarrow \mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$

$$\begin{array}{ccc} \mathrm{DLF}_N & \longrightarrow & \mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim} \\ & \searrow \quad \circlearrowleft \quad \nearrow & \\ & \widehat{\mathrm{DLF}_N} & \end{array}$$

Conjecture (Robert–Wagner): $\widehat{\mathrm{DLF}_N} \longrightarrow \mathbb{S}\mathbb{B}\mathbb{S}\mathrm{Bim}$ is an equivalence of 2-categories.

5. Hochschild homology

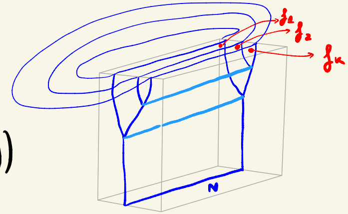
Take Γ braid-like web: $\underline{k} \to \underline{k}$ and let $\hat{\Gamma}$ be its closure. Let $I \subset S_n$ correspond to $\underline{k}$.

Define $\mathcal{F}^T_\infty(\hat{\Gamma}) = \{\text{vinyl } \hat{\Gamma}\text{-foams}\}/\infty\text{-equivalence}$

Proposition (Robert–Wagner): $HH_0(R^I, \mathcal{F}^D_\infty(\Gamma)) \cong \mathcal{F}^T_\infty(\hat{\Gamma})$

Sketch: "Closing up" gives a map $\pi : \mathcal{F}^D_\infty(\Gamma) \to \mathcal{F}^T_\infty(\hat{\Gamma})$.

Now this identifies the left and right R^I actions:



So π factors through $\mathcal{F}^D_\infty(\Gamma) / [\mathcal{F}^D_\infty(\Gamma), R^I] = HH_0(\mathcal{F}^D_\infty(\Gamma))$

Surjectivity comes from the relations. Then dimension count. $\square$

Remark: Upcoming work by Khovanov–Robert–Wagner contains a complete description of all HH_*.

6. Evaluation of vinyl foams

Finally, we categorify the symmetric MOY calculus

Write $A_k = \underbrace{\mathbb{Q}[x_1,\dots,x_N]}_{R}[y_1,\dots,y_k]^{S_k}$

$$J_k = \left\langle \prod_{i=1}^{N}(x_j - y_i) \; : \; j \in \{1,\dots,N\} \right\rangle$$

$$M_{N,k} = A_k / A_k \cap J_k$$

$\sum_{\lambda' \text{ perm of } \lambda} x^{\lambda'}$

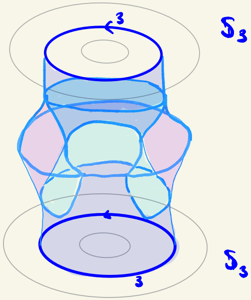
Fact: • $M_{N,k}$ is free over R, with basis $\{ m_\lambda(y_1,\dots,y_k) : \lambda$ is a Young tableaux with $\leq k$ rows, $\leq N-1$ columns $\}$

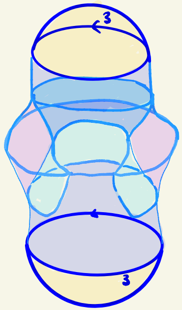
In fact, $\mathrm{grk}(M_{N,k}) = \begin{bmatrix} k+N-1 \\ k \end{bmatrix}$

• $M_{N,k}$ is a commutative Frobenius algebra, with trace

$$\varepsilon_{N,k} : m_\lambda \longmapsto \begin{cases} 1 & \text{if } \lambda = \left.\underbrace{\boxed{\boxplus}}_{N-1}\right\}k \\ 0 & \text{o/w} \end{cases}$$

Evaluation of $\mathcal{S}_K$ - vinyl foam - $\mathcal{S}_K$

$F =$ 

$\rightsquigarrow \Big\langle$ 

$\Big\rangle_K \in A_K \xrightarrow{\pi} M_{N,K} \overset{\varepsilon_{N,K}}{\rightsquigarrow} \langle\langle F \rangle\rangle \in M_{N,K}$

(exterior)

Universal construction: Functor $\mathcal{S}_{K,N} : TLF_N \longrightarrow \mathbb{Z}[x_1,\dots,x_N]$-mod

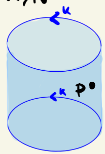
Vinyl graph $\Gamma \mapsto \mathbb{Z}[X] \cdot \{\text{vinyl foams} : \mathcal{S}_K \to \Gamma\} / \langle\langle , \rangle\rangle_N$

Vinyl foam $F \mapsto$ induced homomorphism

Example: $\mathcal{S}_{k,N}(\mathbb{S}_k) = M_{N,k}$, a categorification of $\langle\langle \bigcirc \rangle\rangle_N = \begin{bmatrix} k+N-1 \\ k \end{bmatrix}$

Sketch of proof: let $\varphi : A_k \longrightarrow \mathcal{S}_{k,N}(\mathbb{S}_k)$

$$P \longmapsto$$



Using the evaluation, one shows that these actually span $\mathcal{S}_{k,N}(\mathbb{S}_k)$

Recall also that the evaluation kills $J_k \cap A_k$, so we get an induced map

$$\tilde{\varphi} : M_{N,k} \longrightarrow \mathcal{S}_{k,N}(\mathbb{S}_k)$$

We show that it is injective: take $x \in M_{N,k}$ nonzero. Since $M_{N,k}$ is a Frob. alg, there is some $y \in M_{N,k}$ such that $\varepsilon(xy) \neq 0$. Let $\tilde{\varphi}(x) = X$, $\varphi(y) = Y$ be R_N-linear combinations of vinyl $\mathbb{S}_k$-foams-$\mathbb{S}_k$

Then $\langle\langle X \circ Y \rangle\rangle_N = \varepsilon(xy) \neq 0$ so $X = \tilde{\varphi}(x) \neq 0$. $\square$

Theorem (Robert-Wagner): $\operatorname{grk}(\mathcal{S}_{k,N}(\Gamma)) = \langle\langle \Gamma \rangle\rangle_N$.

Closing remarks

- An earlier (satisfactory) approach (2014) to N-foams exists: Queffelec-Rose "KhR via categorical skew Howe duality"
- Robert-Wagner's approach is "backwards compatible". Also, their technology describes not only 2-morphisms in $\mathcal{S}$SBim, but also the 1-morphisms (the SBims themselves).
- The invariance of Robert-Wagner's symmetric KhR is proved by finding a spectral sequence from HHH $\Rightarrow$ SymKhR.
- Much to do: Deformations, integrality, categorical actions, spectral sequences, other RT invariants...
- Fundamental open question: is there a way to define symmetric Khovanov-Rozansky homology for webs in general? (As opposed to braid-like webs only)

Thank you!

Questions?