The deal with foams

Today: intro to $\mathfrak{sl}_n$-foams

Next time: HOMFLY-PT foams, approach to Hochschild homology, etc.

1. How to get a link invariant from a braided category.

Let $\mathcal{C}$ be a monoidal category with duals (e.g. $\mathcal{C} = \mathrm{Rep}^{f.d.}_{\mathfrak{g}}$, or $\mathcal{C} = A\text{-mod}$, A Hopf algebra)

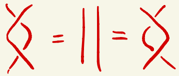
Example: let A be a Hopf algebra (bialgebra + antipode), such as $U(\mathfrak{g})$ with

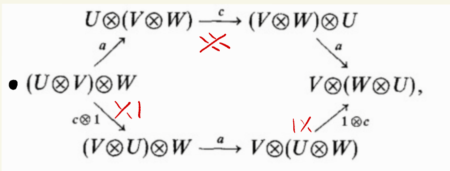
- 1
- $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$
- $\varepsilon(x) = 0$, $x \in \mathfrak{g}$
- $S(x) = -x$, $x \in \mathfrak{g}$

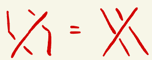
We say that $\mathcal{C}$ is braided if it has a 2-morphism $c : \otimes \to \otimes^{op}$ s.t.:

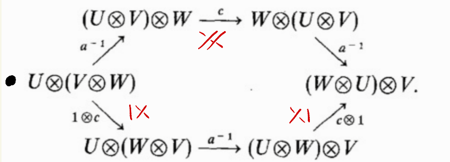
- $c_{XY} : X \otimes Y \to Y \otimes X$ invertible, inverse c_{XY}^{-1}
- Depict c_{XY} as 

and c_{XY}^{-1} as 

so 

- 

$\leftrightarrow$ 

- 

$\leftrightarrow$ 

Note: this introduction is not chronological

Q: how to make A-mod braided? (A Hopf algebra)

$c_{XY} := X \otimes Y \longrightarrow Y \otimes X$

$X \otimes Y \xrightarrow{R\cdot} X \otimes Y \xrightarrow{\text{swap}} Y \otimes X$, where $R \in A \otimes A$

Conditions on R:
$\text{swap} \circ \Delta(a) = R\Delta(a)R^{-1}$
$(\Delta \otimes id)(R) = R_{13}R_{23}$
$(id \otimes \Delta)(R) = R_{13}R_{12}$

"Universal R-matrix"

Key example: $A = U_q(\mathfrak{g})$, $R = \sum_{a_i \text{ basis of } U(\mathfrak{b}_+)} a_i \otimes a_i^*$. Such Hopf algebras are called quasitriangular.

(technically infinite, take completion and restrict...)

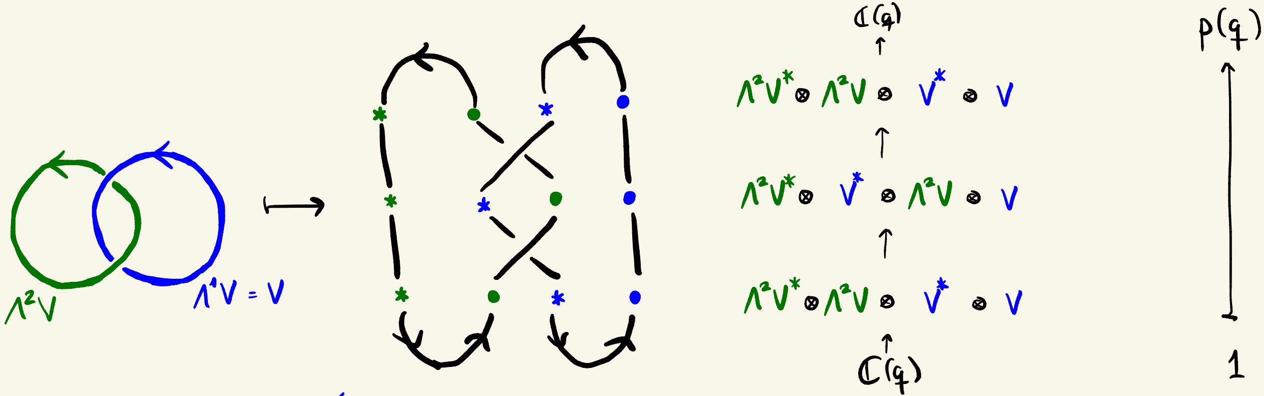
Theorem (Reshetikhin–Turaev, 1990):

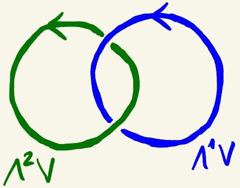
Braided monoidal categories with duals yield invariants of colored oriented links. (technically framed...)

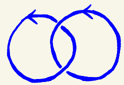
colored → by objects in $\mathcal{C}$

Construction (Reshetikhin–Turaev):

Example: Take the full subcategory of $U_q(gl_n)$-mod $\otimes$-generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$



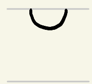
So P_{gl_n} () $= p(q)$

If we color every component by V, we get HOMFLY() $(q, t = q^N)$

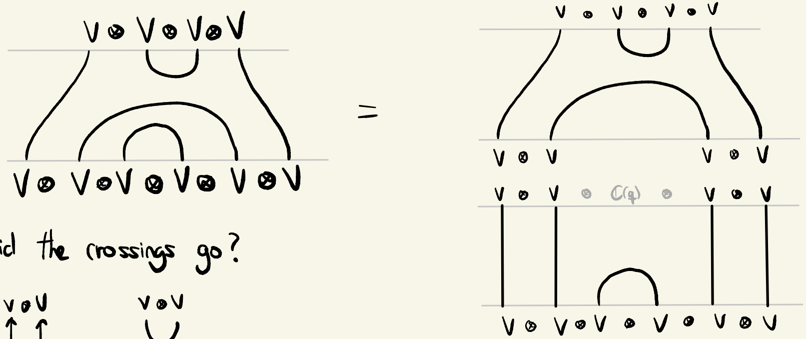
2. Webs (MOY calculus)

For some categories, hom spaces have nice combinatorial bases.

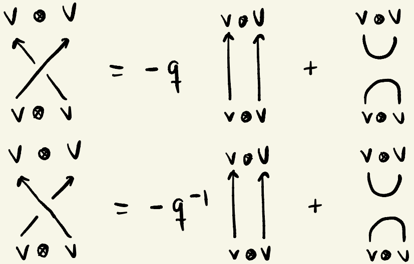
Example 1: $\mathcal{C} \subseteq U_q(\mathfrak{sl}_2)$ full subcategory $\otimes$-generated by V $(= \mathbb{C}(q)^2)$

Then the morphisms  $\mathbb{C}(q) \leftarrow V \otimes V$ ($=\Lambda^2 V$), $q^{-\delta_{j<i}} e_i \wedge e_j \leftarrow e_i \otimes e_j$ and  $V \otimes V \leftarrow \mathbb{C}(q)$, $q e_1 \otimes e_1 + e_2 \otimes e_2 \leftarrow 1$

$\otimes, \circ$ - generate all Hom spaces, i.e. all morphisms are linear combinations of things like:



Wait... where did the crossings go?

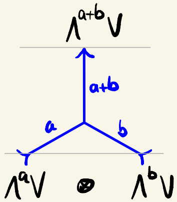


Rmk 1: these diagrams are the morphisms of the Temperley-Lieb category for $\delta = -[2]$

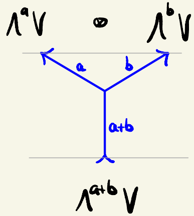
Rmk 2: $Kar(\mathcal{C}) = U_q(\mathfrak{sl}_n)$-mod so morally this controls all of it

Example 2: Take again $\mathcal{C}$ = full subcategory of $U_q(\mathfrak{gl}_n)$ $\otimes$-generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

Then every morphism is $\otimes, \circ$ -generated by:

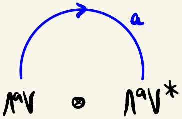


$$e_I \otimes e_J \mapsto q^{-|J<I|} e_{I \sqcup J} \qquad I \cap J = \emptyset$$

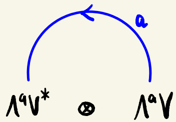


$$e_K \mapsto \sum_{I \sqcup J = K} q^{-|I<J|} e_I \otimes e_J$$

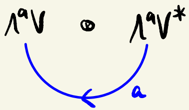
$\mathbb{C}(q)$ $\qquad$ $\mathbb{C}(q)$



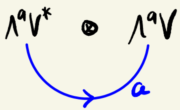
$$e_I \otimes e_J^* \mapsto q^{|I<I_N| - |I_N<I|} \delta_{IJ}$$



$$e_I^* \otimes e_J \mapsto e_I^*(e_J)$$



$$1 \mapsto \sum_{|I|=a} e_I \otimes e_I^*$$

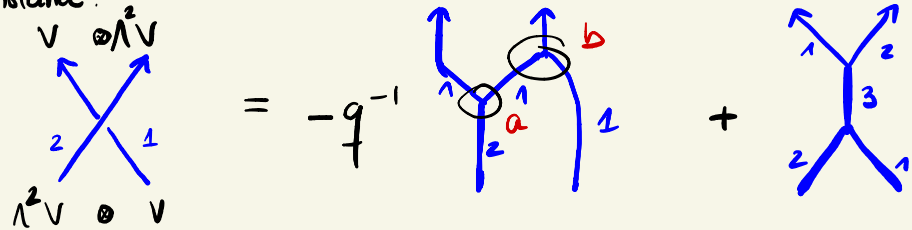


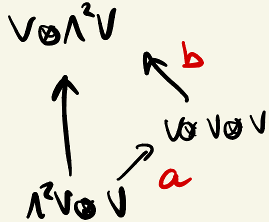
$$1 \mapsto \sum_{|I|=a} q^{-|I_N<I| + |I<I_N|} e_I \otimes e_I^*$$

$\mathbb{C}(q)$ $\qquad$ $\mathbb{C}(q)$

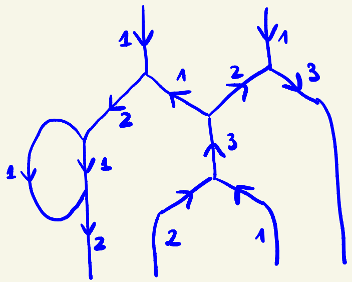
Example 2 (continued): Take again $\mathcal{C}$ = full subcategory of $U_q(\mathfrak{gl}_n)$ $\otimes$-generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

For instance:

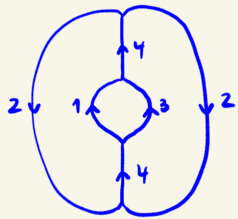




Such combinations are called webs, or MOY graphs:

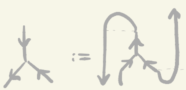


A web (= MOY graph)



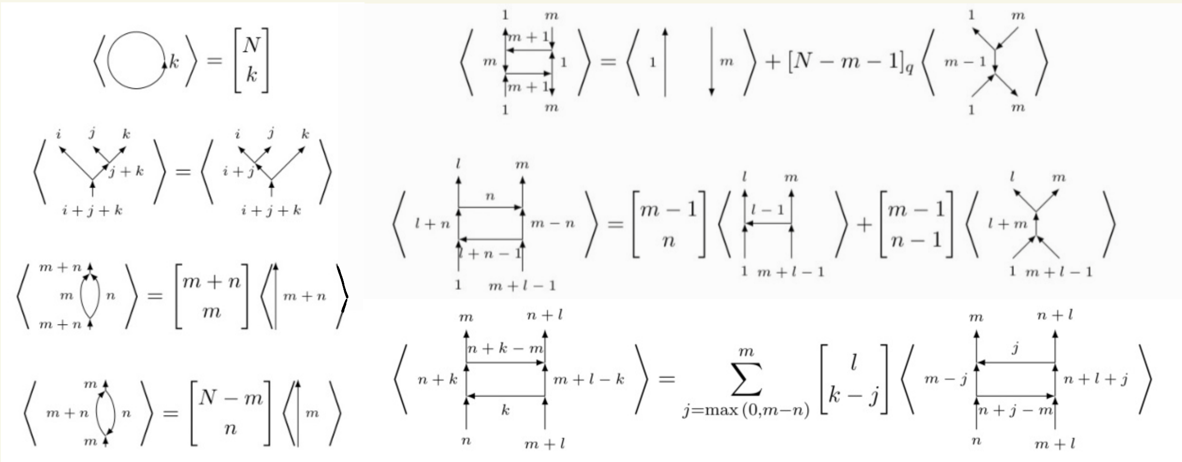
A closed web

Note:



Example 2 (continued): Take again $\mathcal{C}$ = full subcategory of $U_q(\mathfrak{gl}_n)$ $\otimes$-generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

We have more relations:



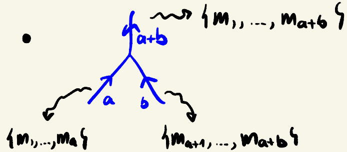
Theorem (Murakami–Ohtsuki–Yamada, 1998): these relations suffice to evaluate any closed web.

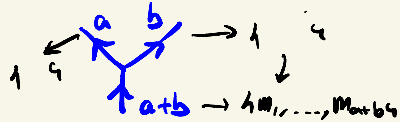
Theorem (Cautis–Kamnitzer–Morrison, 2014): the functor $\mathbf{Web}_n \to \mathcal{C}$ is in fact an equivalence.

Coloring formula for web evaluation:

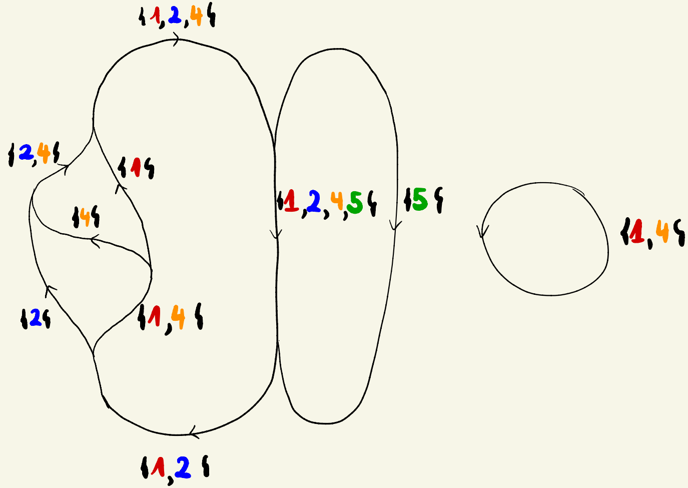
Let Γ be a closed web.

__Definition:__ A coloring of Γ is an assignment of the edges to subsets of $\{1, 2, \dots, n\}$ such that:

- $\Big|_k \rightsquigarrow \{m_1, \dots, m_k\}$
- 

- 

Example:



Let c be a coloring. For a pair of pigments $i,j \in \{1,2,3,\dots,n\}$, let

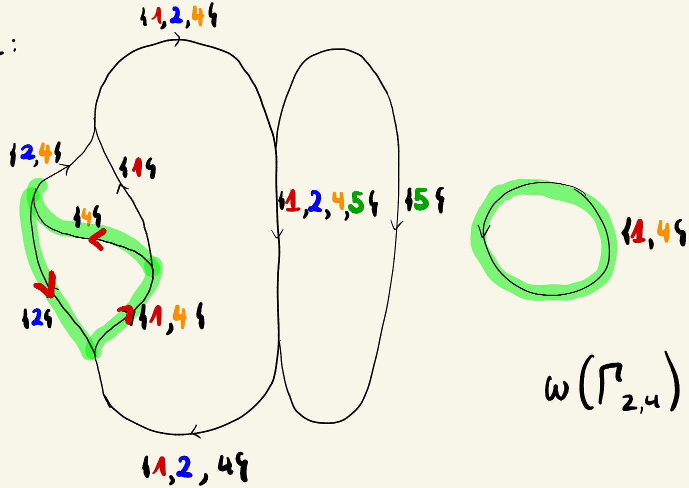
$$\Gamma_{ij}(c) = \{\text{edges colored by } j\} \,\Delta\, \{\text{edges colored by } i, \text{ reversed}\}$$

Define $w(\Gamma_{ij}) :=$ signed # circles in Γ_{ij}

$$w(c) = \sum_{1 \le i < j \le n} w(\Gamma_{ij})$$

Write $\langle \Gamma, c \rangle = q^{w(c)}$

In the example:



$w(\Gamma_{2,4}) = 2$

$\langle \Gamma, c \rangle = \sum q$

q^2

Theorem (coloring formula for web evaluation)

$$\langle \Gamma \rangle = \sum_{c \in col(\Gamma)} \langle \Gamma, c \rangle$$

Corollary: $\langle \Gamma \rangle \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$.

Clear for knots, not so for webs

Categorification?

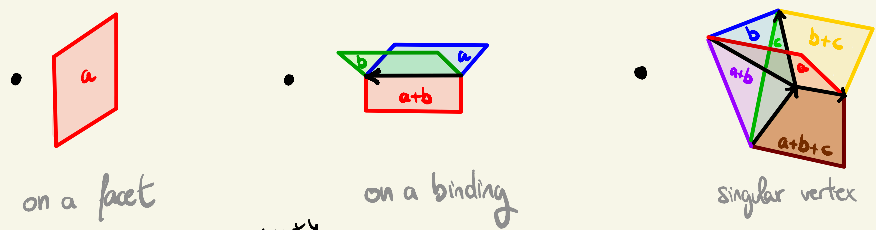
Specifically: $L \to H_*(L)$ s.t.

$$\langle L \rangle = \sum_i (-1)^i \, \mathrm{qdim}\, H_i(L)$$??

3. Foams (via Robert-Wagner evaluation)

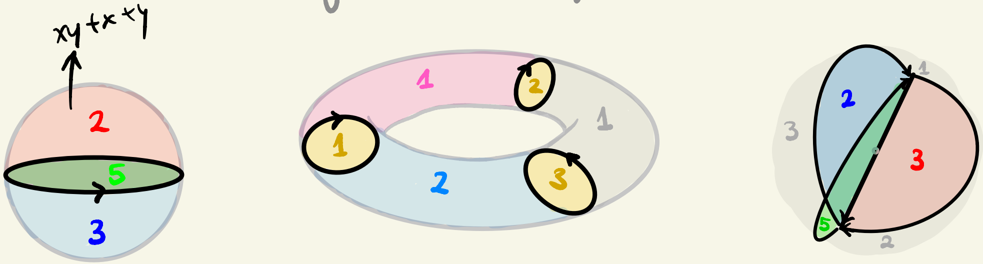
Foams can be thought of as cobordisms of webs. More precisely:

A $\mathfrak{gl}_n$-foam is a finite 2-dimensional CW complex embedded in $\mathbb{R}^3$ with a thickness function $\{2\text{-cells in } F\} \to \{1, \dots, n\}$ such that the neighborhood of a point is homeomorphic to one of the following:



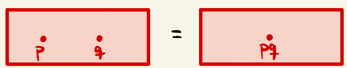
+ orientation condition
facets ↔ bindings

Examples:



Remark: facets may have "decorations", symmetric polynomials in (thickness)-many variables.

They behave multiplicatively:



Some motivation for what follows:

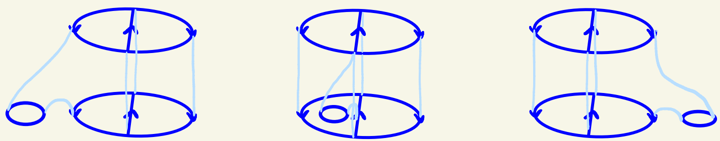
To a link we will associate a complex $C_*(\text{link})$ of graded modules over $\mathbb{Z}[x_1,\dots,x_n]^{S_n}$

Inspiration comes from the sl_3 case, worked out by Khovanov. Khovanov's choices were in turn inspired by the sl_2 case (Khovanov homology).

- sl_2 : $C(\text{unlink}) = \text{TQFT}(\text{unlink}) = \mathbb{Z}[x]/(x^2) = H^*(\mathbb{CP}^1)$ after shift, graded rank is $q^{-1}+q = [2]$ (deg 2)
- sl_3 : $C(\text{unlink})$? It should have graded rank $[3]$ so take $H^*(\mathbb{CP}^2) = \mathbb{Z}[x]/(x^3)$

$C(\ominus)$? • It should have graded rank $\langle \ominus \rangle_n = [3]\cdot[2]$
 • It should have three $\mathbb{Z}[x]/(x^3)$-module structures:

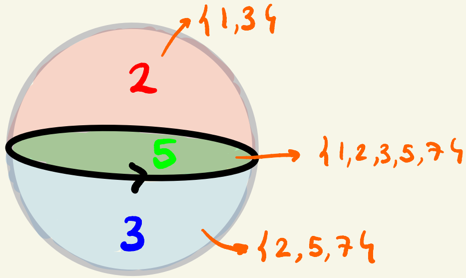
Take $H^*(Fl_3)$



$$H^*(Fl_3) = \mathbb{Z}[x_1,x_2,x_3] \Big/ (x_1+x_2+x_3,\ x_1x_2+x_2x_3+x_1x_3,\ x_1x_2x_3)$$

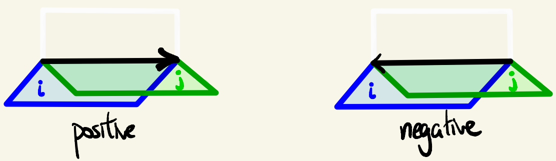
Foam evaluation provides the "denominators" of these graded modules.

Define colorings as with webs:



Observations:

- $F_i = \bigcup$ facets containing i is an oriented surface
- $F_{ij} = \bigcup$ facets containing i XOR j is an oriented surface (take orientations of the facets containing i reversed)
- $F_{ij} = F_{ij} \cap F_i \sqcup F_{ij} \cap F_j$, and these meet at circles which inherit an orientation:



Denote θ^+_{ij} for the number of positive circles in F_{ij}

Degree of a foam:

Choose any coloring c and define $\chi_n(F) = \sum_{1 \le i < j \le n} \chi(F_{ij})$. "Weighted Euler characteristic"

Define the degree of F as $\deg(F) = -\chi_n(F) + 2 \sum_{\text{facets } f} \deg(P_f)$

↳ decoration on f

Q

Foam evaluation

Foam $F \rightsquigarrow \langle F \rangle \in \mathbb{Z}[x_1, \dots, x_n]^{S_n}$

Robert and Wagner's idea: take inspiration from the coloring formula for web evaluation.

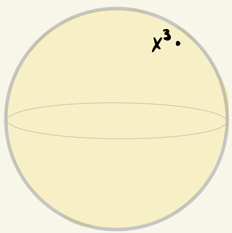
Denote $X = \{x_1, \dots, x_n\}$ and given $S \subset \{1, \dots, n\}$, $X_S = \{x_i : i \in S\}$.

Define $$\langle F, c \rangle = \frac{(-1)^{s(F,c)} \prod_{f \text{ facet}} P_f(X_{c(f)})}{\prod_{i<j} (x_i - x_j)^{\chi(F_{ij}(c))/2}}$$

Remark: $s(F,c)$ is given by $\sum_{i=1}^{n} i \frac{\chi(F_i(c))}{2} + \sum_{1 \le i < j \le n} \theta^+_{ij}(F,c)$

and finally: $$\langle F \rangle = \sum_{c \in col(F)} \langle F, c \rangle$$

Example: $n=3$,

$F =$ 

thickness 1

$c = \{1\} \Rightarrow \langle F, c \rangle = \frac{-x_1^3}{(x_1 - x_2)(x_1 - x_3)}$ $\quad s(F,c) = 1$

$c = \{2\} \Rightarrow \langle F, c \rangle = \frac{x_2^3}{(x_1 - x_2)(x_2 - x_3)}$ $\quad s(F,c) = 2$

$c = \{3\} \Rightarrow \langle F, c \rangle = \frac{-x_3^3}{(x_1 - x_3)(x_2 - x_3)}$ $\quad s(F,c) = 3$

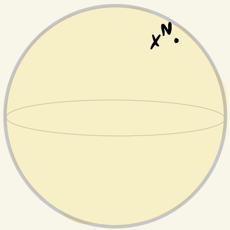
$$\langle F \rangle = \frac{-x_1^3(x_2 - x_3) + x_2^3(x_1 - x_3) - x_3^3(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$\langle F \rangle = \frac{-x_1^3(x_2-x_3) + x_2^3(x_1-x_3) - x_3^3(x_1-x_2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}$$

$$= \frac{-x_1^3x_2 + x_1x_2^3 - x_2^3x_3 + x_3x_2^3 - x_3^3x_1 + x_1x_3^3}{x_1^2x_2 - x_1^2x_3 + x_2^2x_3 - x_2x_3^2 + x_3^2x_1 - x_3x_1^2}$$

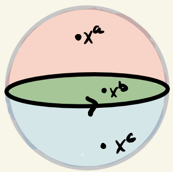
$$= -(x_1+x_2+x_3) \qquad (!)$$

Example:

$\Big\langle$  $\Big\rangle$

$$= \frac{-x_1^N(x_2-x_3) + x_2^N(x_1-x_3) - x_3^N(x_1-x_2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}$$

$$= \frac{-\det\begin{pmatrix} x_1^N & x_1 & 1 \\ x_2^N & x_2 & 1 \\ x_3^N & x_3 & 1 \end{pmatrix}}{\text{Vandermonde}} = -S_{n-2,0,0}(x_1,x_2,x_3) = -\sum_{i+j+k=n-2} x_1^i x_2^j x_3^k$$

Remark: $\Big\langle$  $\Big\rangle = S_{a-1,b-1,c}$, among many combinatorial relations.

It is almost immediate that $\langle F \rangle$ is a symmetric rational function

Fact: $\langle F \rangle$ is in fact a polynomial. Proof relatively painless but omitted.

4. A foamy link homology

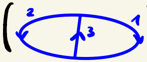
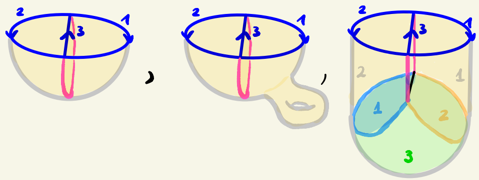
We first define a functor $\mathcal{F} : \mathrm{Foam}_n \longrightarrow \mathbb{Z}[X]\text{-gmod}$

(webs, foams between them) $\qquad \mathbb{Z}[x_1,\dots,x_n]$, $\deg x_i = 2$.

<u>On objects</u>:

$$\Gamma \longmapsto W_n(\Gamma) / \mathrm{Ker}((\cdot,\cdot)_n)$$

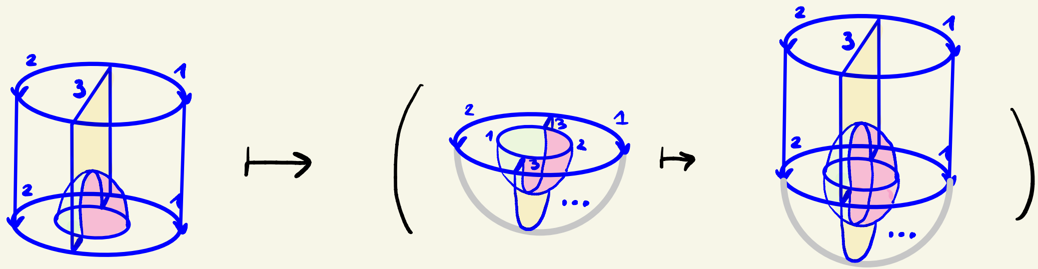
where: $W_n(\Gamma)$ = free module with basis foams with boundary $\emptyset \to \Gamma$

$W_n($  $) = \mathbb{Z}[X]\cdot\{$  $, \ldots \}$

and $(F,G)_n = \langle \bar{G} \circ F \rangle$

<u>On morphisms:</u> $F : \Gamma_1 \to \Gamma_2$ gives $W_n(\Gamma_1) \to W_n(\Gamma_2)$

$$G \longmapsto F \circ G$$

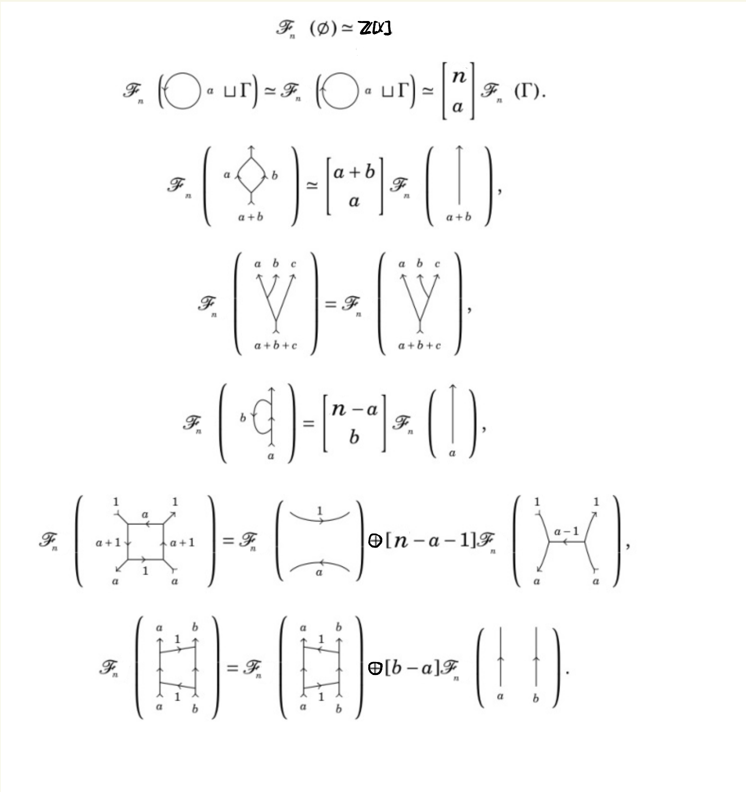


Note if $L \in W_n(\Gamma_1)$ is in $\mathrm{Ker}(\langle\cdot,\cdot\rangle_n)$ then $F \circ L \in \mathrm{Ker}(\langle\cdot,\cdot\rangle_n)$

This is the so-called universal construction. → Works in much greater generality.

Theorem: $\mathcal{F}_n(\Gamma)$ is a free graded $\mathbb{Z}[X]$-module of graded rank $\langle \Gamma \rangle$.

Proof: We denote grading shifts as: $q^a \cdot M = M(a)$. One shows using the evaluation that:



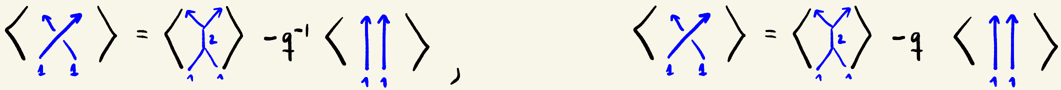
Now any web reduces to $\emptyset$ using finitely many of these relations. At each step we obtain a projective module.

Projective modules over $\mathbb{Z}[X]$ are free.

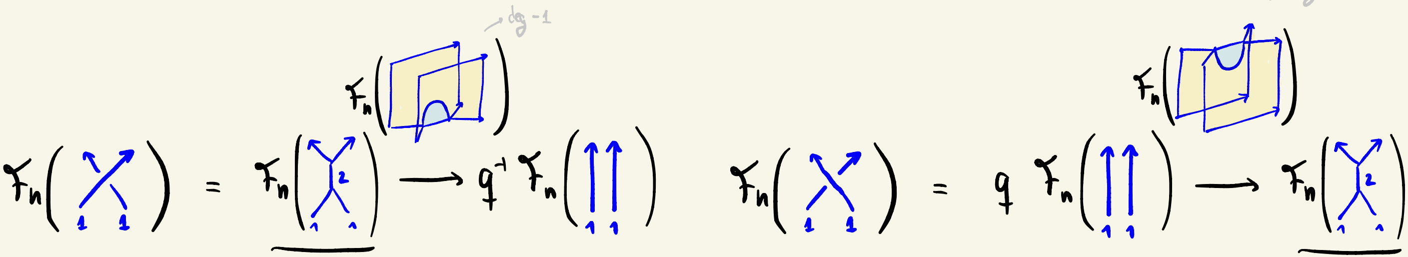
The relations are those for web evaluation so we get $\mathrm{grk}(\mathcal{F}(\Gamma)) = \langle \Gamma \rangle$. $\square$

What about the knots ?

Recall: we can rewrite the crossings as:



As in Khovanov homology, we categorify this as complexes, using $\mathcal{F}_n$ as a sort of TQFT:



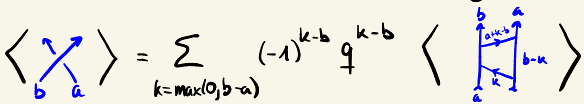
So the complex associated to an oriented link diagram is

$$C(L) = \bigotimes_{\text{crossings } x} (\text{Complex at } x)$$

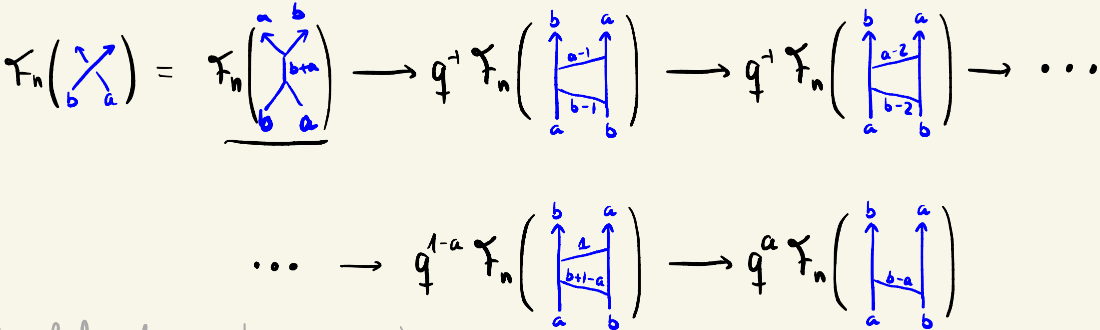
Theorem: This is an invariant of oriented links that categorifies the sl_n link polynomial.
(up to overall grading shift) (It's considered up to homotopy)

A generalization for free

In fact we can express <u>any</u> crossing as:



So we can form the more general Rickard complex:



(Similarly for negative crossings)

$$C_*(L) = \bigotimes_{\text{crossings } x} (\text{Complex at } x)$$

Theorem: This is an invariant of <u>colored</u> oriented links that categorifies the colored sl_n link polynomial.
(up to overall grading shift) (It's considered up to homotopy)

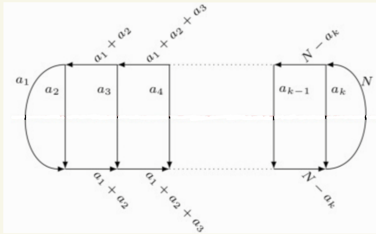
5. A cool application

Fact: $H_T^*(Fl_n) = \dfrac{\mathbb{Z}[T_i, X_i]}{P(X_i) - P(T_i),\ P \text{ sym poly}}$ and $H_T^*(Fl_n(a_1,\dots,a_n)) = H_T^*(Fl_n)^{S_{a_1}\times\dots\times S_{a_n}}$

$\mathbb{C}^{a_1} \subseteq \mathbb{C}^{a_1+a_2} \subseteq \dots \mathbb{C}^{a_1+\dots+a_n}$

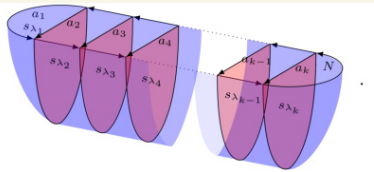
Proposition (Robert-Wagner):

$$H_T^*(Fl_n(a_1,\dots,a_n)) \cong \mathcal{F}_n \left(\quad \right)$$



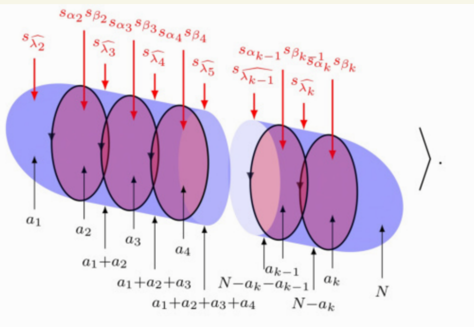
The isomorphism is given by:

$$\prod_{i=1}^k s_{\lambda_i}(X_{a_i+1},\dots,X_{a_{i+1}}) \mapsto$$



This can be used to compute the structure constants for $H_T^*(Fl_n(a_1,\dots,a_n))$ as a ring:

$$\mathbf{c}_{\alpha\beta}^{\lambda} \stackrel{def}{=} (-1)^{\sum_{j=2}^k |\widehat{\lambda_j}| + N(N+1)/2} \left\langle \quad \right\rangle.$$



Thank you!

Questions?