

The deal with foams

Today: intro to \mathfrak{sl}_n -foams

Next time: HOMFLY-PT foams, approach to Hochschild homology, etc.

1. How to get a link invariant from a braided category.

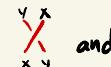
Let \mathcal{C} be a monoidal category with duals (e.g. $\mathcal{C} = \text{Rep}_\mathbb{Z}^{\text{fd}}$, or $\mathcal{C} = A\text{-mod}$, A Hopf algebra)

Example: let A be a Hopf algebra (bialgebra + antipode), such as $U(g)$

$$\begin{array}{lll} i & \Delta(x) = x \otimes 1 + 1 \otimes x & x \in g \\ & \varepsilon(x) = 0 & x \in g \\ & S(x) = -x & x \in g \end{array}$$

We say that \mathcal{C} is braided if it has a 2-morphism $c: \otimes \rightarrow \otimes^\text{op}$ s.t.:

- $c_{xy}: X \otimes Y \rightarrow Y \otimes X$ invertible, inverse c_{xy}^{-1}

- Depict c_{xy} as  and c_{xy}^{-1} as  so  =  = 

$$U \otimes (V \otimes W) \xrightarrow{c} (V \otimes W) \otimes U$$

$$\swarrow a \qquad \qquad \searrow a$$

$$(U \otimes V) \otimes W \qquad \qquad V \otimes (W \otimes U),$$

$$\begin{matrix} c \otimes 1 & \swarrow \\ \text{---} & \end{matrix} \quad \begin{matrix} 1 \otimes c & \nearrow \\ \text{---} & \end{matrix}$$

$$(V \otimes U) \otimes W \xrightarrow{a} V \otimes (U \otimes W)$$

$$\leftrightarrow \text{---} = \text{---}$$

$$(U \otimes V) \otimes W \xrightarrow{c} W \otimes (U \otimes V)$$

$$\swarrow a^{-1} \qquad \qquad \searrow a^{-1}$$

$$\leftrightarrow \text{---} = \text{---}$$

$$U \otimes (V \otimes W) \qquad \qquad (W \otimes U) \otimes V,$$

$$\begin{matrix} 1 \otimes c & \swarrow \\ \text{---} & \end{matrix} \quad \begin{matrix} c \otimes 1 & \nearrow \\ \text{---} & \end{matrix}$$

$$U \otimes (W \otimes V) \xrightarrow{a^{-1}} (U \otimes W) \otimes V$$

Note: this introduction is not chronological

Q: how to make $A\text{-mod}$ braided? (A Hopf algebra)

$$c_{xy} := X \otimes Y \rightarrow Y \otimes X$$

Conditions on R :

$$\begin{aligned} \text{swap} \circ \Delta(a) &= R \Delta(a) R^{-1} && \text{"Universal R-matrix"} \\ (\Delta \otimes \text{id})(R) &= R_{13} R_{23} \\ (\text{id} \otimes \Delta)(R) &= R_{13} R_{12} \end{aligned}$$

Key example: $A = U_q(\mathfrak{g})$, $R = \sum_{a_i \text{ basis of } U(\mathfrak{b}_+)} a_i \otimes a_i^*$. Such Hopf algebras are called quasitriangular.
 (technically infinite, take completion and restrict...)

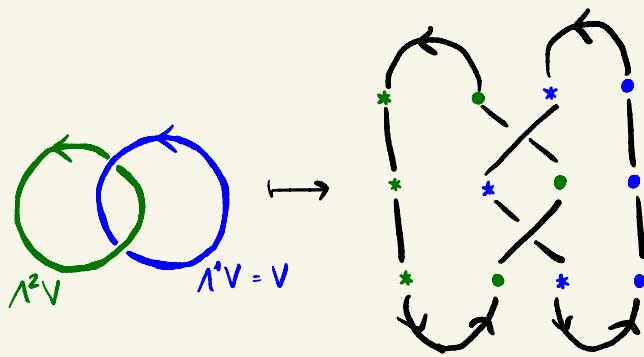
Theorem (Reshetikhin - Turaev, 1990):

Braided monoidal categories with duals yield invariants of colored oriented links. (technically framed...)

↓
by objects in \mathcal{C}

Construction (Reshetikhin-Turaev) :

Example: Take the full subcategory of $U_q(\mathfrak{gl}_n)\text{-mod}$ \otimes -generated by $\Lambda^0 V, \dots, \Lambda^n V$
and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$



$$\begin{array}{c}
 \overset{\mathbb{C}(q)}{\uparrow} \\
 \Lambda^0 V^* \otimes \Lambda^2 V \otimes V^* \otimes V \\
 \uparrow \\
 \Lambda^0 V^* \otimes V^* \otimes \Lambda^2 V \otimes V \\
 \uparrow \\
 \Lambda^0 V^* \otimes \Lambda^2 V \otimes V^* \otimes V \\
 \downarrow \quad \overset{\mathbb{C}(q)}{\uparrow} \\
 1 \qquad p(q)
 \end{array}$$

$$\text{So } P_{\mathfrak{gl}_n} \left(\text{link diagram} \right) = p(q)$$

If we color every component by V , we get HOMFLY $(\text{link}) (q, t=q^n)$

2. Webs (Moy calculus)

For some categories, hom spaces have nice combinatorial bases.

Example 1: $\mathcal{C} \subseteq U_q(\mathfrak{sl}_2)$ full subcategory \otimes -generated by V ($= C(q)$)

Then the morphisms

$$\begin{array}{ccc} & C(q) & \xrightarrow{\sim} V \\ & \uparrow & q^{-\text{fix.}, \text{nej}} \\ V \otimes V & & e_i \otimes e_j \end{array}$$

and

$$\begin{array}{ccc} & V \otimes V & \xrightarrow{\sim} q e_i \otimes e_j + e_i \otimes e_j \\ & \uparrow & \downarrow \\ & C(q) & 1 \end{array}$$

\otimes, \circ - generate all Hom spaces, i.e. all morphisms are linear combinations of things like:

$$\begin{array}{c} V \otimes V \otimes V \otimes V \\ \diagup \quad \diagdown \\ \text{---} = \text{---} \\ \diagdown \quad \diagup \\ V \otimes V \otimes V \otimes V \otimes V \otimes V \end{array}$$

Wait... where did the crossings go?

$$\begin{array}{lcl} \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} & = -q & \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} \\ \\ \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} & = -q^{-1} & \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ V \otimes V \\ \diagdown \quad \diagup \\ V \otimes V \end{array} \end{array}$$

Rank 1: these diagrams are the morphisms of the Temperley-Lieb category for $\delta = -22$

Rank 2: $\text{Kar}(\mathcal{C}) = U_q(\mathfrak{sl}_n) \text{-mod}$ so morally this controls all of it

Example 2: Take again \mathcal{C} = full subcategory of $\mathbf{U}_q(\mathfrak{gl}_n)$ \otimes -generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

Then every morphism is \otimes, \circ -generated by:

$$\begin{array}{c} \Lambda^{a+b} V \\ \uparrow \\ \Lambda^a V \otimes \Lambda^b V \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} q^{-|I \times J|} e_{IJ} \\ \uparrow \\ e_I \otimes e_J \quad \text{INJ}=\emptyset \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \Lambda^a V \circ \Lambda^b V \\ \uparrow \\ \Lambda^{a+b} V \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \sum_{I \times J = k} q^{-|I \times J|} e_I \circ e_J \\ \uparrow \\ e_k \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

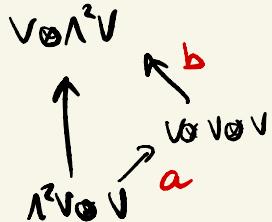
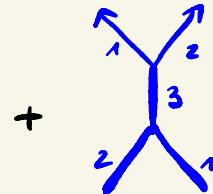
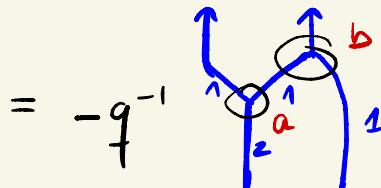
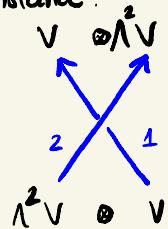
$$\begin{array}{c} \mathbb{C}(q) \quad q^{|I < I_n| - |I_n < I|} \delta_{IJ} \\ \uparrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \Lambda^a V^* \otimes \Lambda^a V \\ \uparrow \\ e_I^* \otimes e_J \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \Lambda^a V^* \circ \Lambda^a V \\ \uparrow \\ e_I^* \otimes e_J \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} e_I^*(e_J) \\ \uparrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$\begin{array}{c} e_I \otimes e_I^* \Lambda^a V \otimes \Lambda^a V^* \\ \uparrow \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \sum_{|I|=a} e_I \otimes e_J^* \\ \uparrow \\ 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \Lambda^a V^* \otimes \Lambda^a V \\ \uparrow \\ e_I \otimes e_J^* \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \sum_{|I|=a} q^{|I_n < I| + |I < I_n|} e_I \otimes e_J^* \\ \uparrow \\ 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

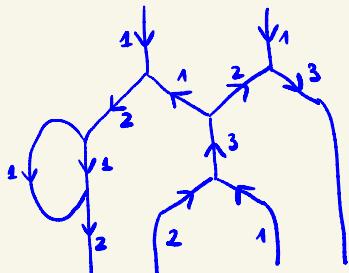
Example 2: Take again \mathcal{C} = full subcategory of $U_q(\mathfrak{gl}_n)$ \otimes -generated by $\Lambda^0 V, \dots, \Lambda^n V$ and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

(continued)

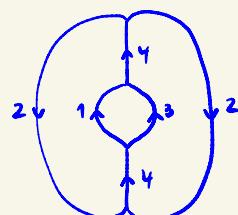
For instance:



Such combinations are called webs, or MOY graphs:



A web (= MOY graph)



A closed web

$\mathbb{C}(q)$



$\mathbb{C}(q)$

Note: =

Example 2: Take again \mathcal{C} = full subcategory of $U_q(\mathfrak{gl}_n)$ \otimes -generated by $\Lambda^0 V, \dots, \Lambda^n V$
 (continued) and their duals $\Lambda^0 V^*, \dots, \Lambda^n V^*$

We have more relations:

$$\begin{aligned}
 \left\langle \bigcirc \xrightarrow{k} \right\rangle &= \binom{N}{k} \\
 \left\langle \begin{array}{c} i & j & k \\ \nearrow & \searrow & \downarrow \\ i+j+k & & \end{array} \right\rangle &= \left\langle \begin{array}{c} i & j & k \\ \nearrow & \searrow & \downarrow \\ i+j+k & & \end{array} \right\rangle \\
 \left\langle \begin{array}{c} m+n \\ m \\ \nearrow \\ m+n \\ m \end{array} \right\rangle &= \binom{m+n}{m} \left\langle \begin{array}{c} m+n \\ m \end{array} \right\rangle \\
 \left\langle \begin{array}{c} m+n \\ m \\ \nearrow \\ m+n \\ m \end{array} \right\rangle &= \binom{N-m}{n} \left\langle \begin{array}{c} m \\ n \end{array} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 \left\langle \begin{array}{c} 1 & m \\ m & \nearrow \\ m & m+1 \\ \downarrow & \nearrow \\ 1 & m \end{array} \right\rangle &= \left\langle \begin{array}{c} 1 \\ \uparrow \\ m \end{array} \right\rangle + [N-m-1]_q \left\langle \begin{array}{c} 1 & m \\ m-1 & \nearrow \\ m & m \end{array} \right\rangle \\
 \left\langle \begin{array}{c} l & m \\ l+n & \nearrow \\ l+n & m-n \\ \downarrow & \nearrow \\ 1 & m+l-1 \end{array} \right\rangle &= \binom{m-1}{n} \left\langle \begin{array}{c} l & m \\ l-1 & \nearrow \\ 1 & m+l-1 \end{array} \right\rangle + \binom{m-1}{n-1} \left\langle \begin{array}{c} l & m \\ l+m & \nearrow \\ 1 & m+l-1 \end{array} \right\rangle \\
 \left\langle \begin{array}{c} m & n+l \\ n+k & \nearrow \\ n+k-m & m+l-k \\ \downarrow & \nearrow \\ n & m+l \end{array} \right\rangle &= \sum_{j=\max(0, m-n)}^m \binom{l}{k-j} \left\langle \begin{array}{c} m & n+l \\ m-j & \nearrow \\ n+j-m & m+l \end{array} \right\rangle
 \end{aligned}$$

Theorem (Murakami - Ohtsuki - Yamada, 1998) : these relations suffice to evaluate any closed web.

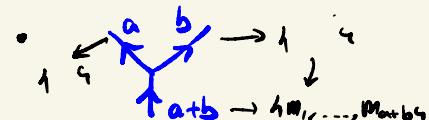
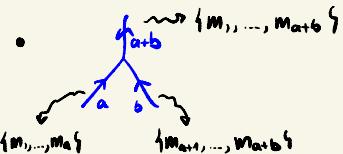
Theorem (Cautis - Kamnitzer - Morrison, 2014) : the functor $\text{Web}_n \rightarrow \mathcal{C}$ is in fact an equivalence.

Coloring formula for web evaluation:

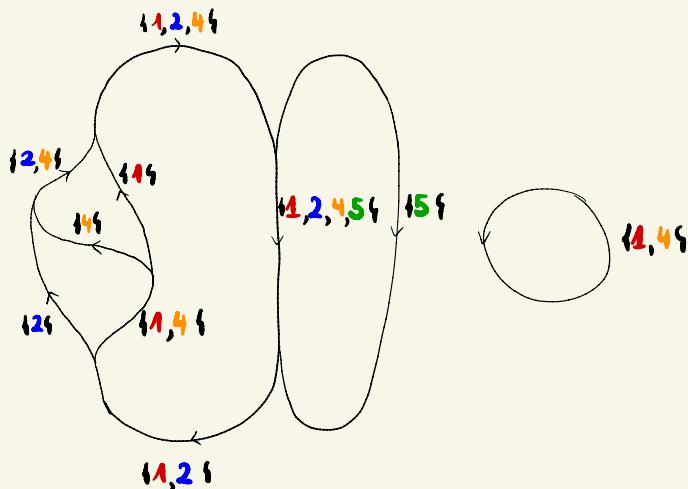
Let Γ be a closed web.

Definition: A coloring of Γ is an assignment of the edges to subsets of $\{1, 2, \dots, n\}$ such that:

- $k \in \{m, \dots, m_{ab}\}$



Example:



Let c be a coloring. For a pair of pigments $i, j \in \{1, 2, 3, \dots, n\}$, let

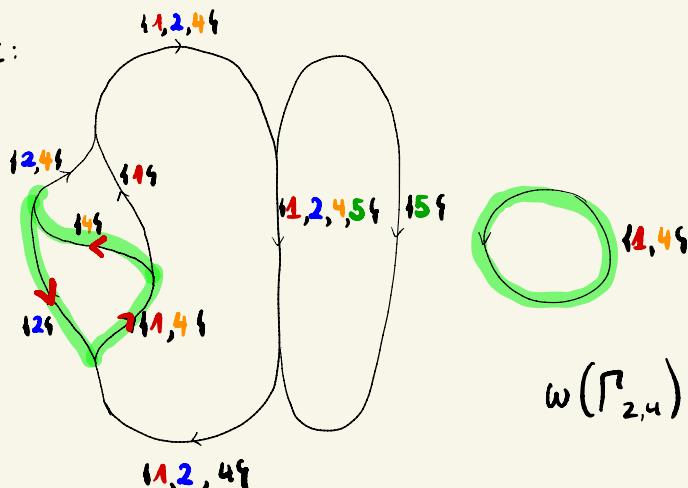
$$\Gamma_{ij}(c) = \{ \text{edges colored by } j \} \Delta \{ \text{edges colored by } i, \text{ reversed} \}$$

Define $w(\Gamma_{ij}) := \text{signed } \# \text{ circles in } \Gamma_{ij}$

$$w(c) = \sum_{1 \leq i < j \leq n} w(\Gamma_{ij})$$

Write $\langle \Gamma, c \rangle = q^{w(c)}$

In the example:



$$w(\Gamma_{2,4}) = 2$$

$$\langle \Gamma, c \rangle = \sum_q q^e$$

Q

Theorem (coloring formula for web evaluation)

$$\langle \Gamma \rangle = \sum_{c \in \text{col}(\Gamma)} \langle \Gamma, c \rangle$$

Corollary: $\langle \Gamma \rangle \in \mathbb{Z}_{\geq 0}[[q, q^{-1}]]$.

Clear for knots,
not so for webs

Categorification?

Specifically: $L \rightarrow H_*(L)$ st.

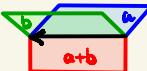
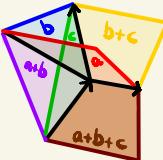
??

$$\langle L \rangle = \sum_i (-1)^i \text{gdim } H_i(L)$$

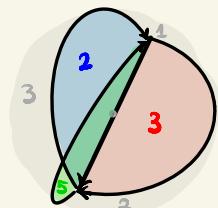
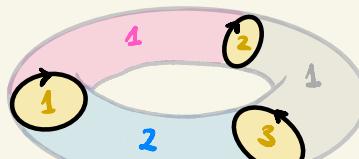
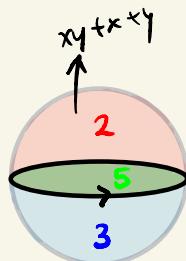
3. Foams (via Robert-Wagner evaluation)

Foams can be thought of as cobordisms of webs. More precisely:

A **g,n -foam** is a finite 2-dimensional CW complex embedded in \mathbb{R}^3 with a thickness function $\{2\text{-cells in } F\} \rightarrow \{1, \dots, n\}$ such that the neighborhood of a point is homeomorphic to one of the following:

-  on a facet
 -  on a binding
 -  singular vertex
- + orientation condition
facets \hookrightarrow bindings

Examples:



Remark: facets may have "decorations", symmetric polynomials in (thickness)-many variables.

They behave multiplicatively:

$$\begin{array}{c} \vdash \\ \vdash \end{array} = \begin{array}{c} \vdash \end{array}$$

Some motivation for what follows:

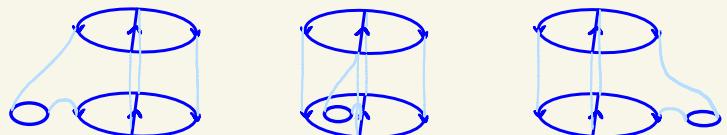
To a link we will associate a complex $C_*(\text{link})$ of graded modules over $\mathbb{Z}[[x_1, \dots, x_n]]^{S_n}$

Inspiration comes from the sl_2 case, worked out by Khovanov. Khovanov's choices were in turn inspired by the sl_2 case (Khovanov homology).

- sl_2 : $C(\text{unlink}) = \text{TQFT}(\text{unlink}) = \mathbb{Z}[x]/_{(x^2)}^{\deg 2} = H^*(CP^1)$ after shift, graded rank is $q^{-1} + q = [2]$

- sl_3 : $C(\text{unlink})?$ It should have graded rank $[3]$ so take $H^*(CP^2) = \mathbb{Z}[x]/_{(x^3)}$
 $C(\bigoplus_i)$? • It should have graded rank $\langle \bigoplus_i \rangle_n = [3] \cdot [2]$

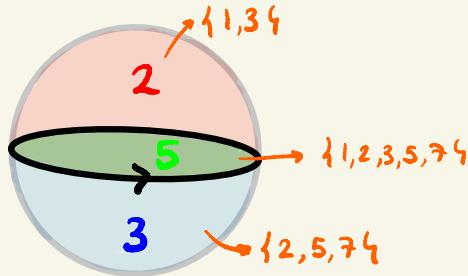
- It should have three $\mathbb{Z}[x]/_{(x^3)}$ -module structures:



$$H^*(Fl_3) = \mathbb{Z}[x_1, x_2, x_3] / (x_1 + x_2 + x_3, x_1 x_2 + x_2 x_3 + x_1 x_3, x_1 x_2 x_3)$$

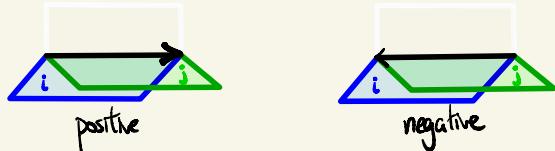
Foam evaluation provides the "denominators" of these graded modules.

Define colorings as with webs:



Observations:

- $F_i = \bigcup$ facets containing i is an oriented surface
- $F_{ij} = \bigcup$ facets containing $i \text{ xor } j$ is an oriented surface (take orientations of the facets containing i reversed)
- $F_{ij} = F_i \cap F_j \sqcup F_{ij} \cap F_j$, and these meet at circles which inherit an orientation:



Denote Θ_{ij}^+ for the number of positive circles in F_{ij}

Degree of a foam:

Choose any coloring c and define $\chi_n(F) = \sum_{i < j \in \text{en}} \chi(F_{ij})$. "Weighted Euler characteristic"

Define the degree of F as $\deg(F) = -\chi_n(F) + 2 \sum_{\text{facets } f} \deg(P_f)$

↳ decoration on f

Q

Foam evaluation

Foam $F \rightsquigarrow \langle F \rangle \in \mathbb{Z}[x_1, \dots, x_n]^{S_n}$

Robert and Wagner's idea: take inspiration from the coloring formula for web evaluation.

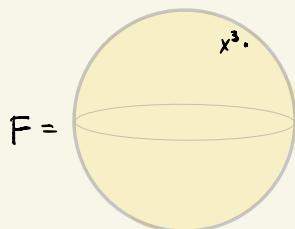
Denote $X = \{x_1, \dots, x_n\}$ and given $S \subset \{1, \dots, n\}$, $X_S = \{x_i : i \in S\}$.

$$\text{Define } \langle F, c \rangle = \frac{(-1)^{s(F, c)} \prod_{\substack{\text{facet} \\ \{i, j\}}} P_{ij}(X_{c(ij)})}{\prod_{i < j} (x_i - x_j)^{x(F_{ij}(c))/2}}$$

Remark: $s(F, c)$ is given by $\sum_{i=1}^n i \frac{x(F_{ii}(c))}{2} + \sum_{1 \leq i < j \leq n} \theta_{ij}^+(F, c)$

$$\text{and finally: } \langle F \rangle = \sum_{c \in \text{col}(F)} \langle F, c \rangle$$

Example: $n=3$,



thickness 1

$$c = \{1\} \Rightarrow \langle F, c \rangle = \frac{-x_1^3}{(x_1 - x_2)(x_1 - x_3)} \quad s(F, c) = 1$$

$$c = \{2\} \Rightarrow \langle F, c \rangle = \frac{x_2^3}{(x_1 - x_2)(x_2 - x_3)} \quad s(F, c) = 2$$

$$c = \{3\} \Rightarrow \langle F, c \rangle = \frac{-x_3^3}{(x_1 - x_3)(x_2 - x_3)} \quad s(F, c) = 3$$

$$\langle F \rangle = \frac{-x_1^3(x_2 - x_3) + x_2^3(x_1 - x_3) - x_3^3(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$\begin{aligned}
 \langle F \rangle &= \frac{-x_1^3(x_2-x_3) + x_2^3(x_1-x_3) - x_3^3(x_1-x_2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \\
 &= \frac{-x_1^3x_2 + x_1x_2^3 - x_2^3x_3 + x_3x_2^3 - x_3^3x_1 + x_1x_3^3}{x_1^2x_2 - x_1^2x_3 + x_2^2x_3 - x_2x_3^2 + x_3^2x_1 - x_3x_1^2} \\
 &= -(x_1+x_2+x_3) \quad (\text{!})
 \end{aligned}$$

Example:

$$\begin{aligned}
 \left\langle \begin{array}{c} x^N \\ \text{---} \\ x^N \end{array} \right\rangle &= \frac{-x_1^N(x_2-x_3) + x_2^N(x_1-x_3) - x_3^N(x_1-x_2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \\
 &= -\det \begin{pmatrix} x_1^N & x_1 & 1 \\ x_2^N & x_2 & 1 \\ x_3^N & x_3 & 1 \end{pmatrix} \\
 &= \frac{\text{Vandermonde}}{-S_{n-2,0,0}(x_1, x_2, x_3)} = -\sum_{i,j,k=0-2} x_1^i x_2^j x_3^k
 \end{aligned}$$

Remark: $\left\langle \begin{array}{c} x^a \\ \text{---} \\ x^b \\ \text{---} \\ x^c \end{array} \right\rangle = S_{a-1, b-1, c}$, among many combinatorial relations.

It is almost immediate that $\langle F \rangle$ is a symmetric rational function

Fact: $\langle F \rangle$ is in fact a polynomial. Proof relatively painless but omitted.

4. A foamy link homology

We first define a functor $\mathcal{F} : \text{Foam}_n \longrightarrow \mathbb{Z}[x] \text{-gmod}$

$$\downarrow \quad \quad \quad \downarrow$$

(webs, foams between them) $\mathbb{Z}[x_1, \dots, x_n], \deg x_i = 2$.

On objects:

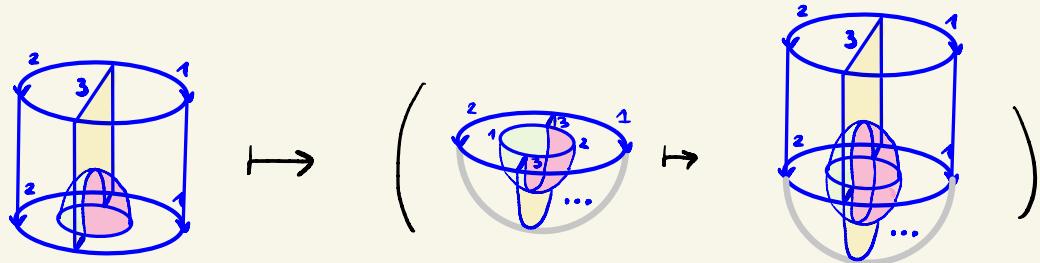
$$\Gamma \longmapsto \frac{W_n(\Gamma)}{\ker((\cdot, \cdot)_n)}$$

where: $W_n(\Gamma) = \text{free module with basis foams with boundary } \emptyset \rightarrow \Gamma$

$$W_n(\text{blue circle}) = \mathbb{Z}[x] \cdot \{ \text{diagrams} \}, \dots \{$$

and $(F, G)_n = \langle \bar{G} \circ F \rangle$

On morphisms: $F : \Gamma_1 \rightarrow \Gamma_2$ gives $W_n(\Gamma_1) \rightarrow W_n(\Gamma_2)$

$$G \longmapsto F \circ G$$


Note if $L \in W_n(\Gamma'_+)$ is in $\text{Ker}((\cdot, \cdot)_n)$ then $F \circ L \in \text{Ker}((\cdot, \cdot)_n)$

This is the so-called **universal construction.** → Works in much greater generality.

Theorem: $\mathcal{F}_n(\Gamma)$ is a free graded $\mathbb{Z}[X]$ -module of graded rank $\langle \Gamma \rangle$.

Proof: We denote grading shifts as: $q^a \cdot M = M(a)$. One shows using the evaluation that:

$$\mathcal{F}_n(\emptyset) \simeq \mathbb{Z}[X]$$

$$\mathcal{F}_n\left(\bigcirc_a \sqcup \Gamma\right) \simeq \mathcal{F}_n\left(\bigcirc_a \sqcup \Gamma\right) \simeq \begin{bmatrix} n \\ a \end{bmatrix} \mathcal{F}_n(\Gamma).$$

$$\mathcal{F}_n\left(\begin{array}{c} a \\ \diagup \quad \diagdown \\ b \\ a+b \end{array}\right) \simeq \begin{bmatrix} a+b \\ a \end{bmatrix} \mathcal{F}_n\left(\begin{array}{c} \\ a+b \end{array}\right),$$

$$\mathcal{F}_n\left(\begin{array}{c} a \quad b \quad c \\ \diagup \quad \diagdown \\ a+b+c \end{array}\right) = \mathcal{F}_n\left(\begin{array}{c} a \quad b \quad c \\ \diagup \quad \diagdown \\ a+b+c \end{array}\right),$$

$$\mathcal{F}_n\left(\begin{array}{c} b \\ \diagup \quad \diagdown \\ a \end{array}\right) = \begin{bmatrix} n-a \\ b \end{bmatrix} \mathcal{F}_n\left(\begin{array}{c} \\ a \end{array}\right),$$

$$\mathcal{F}_n\left(\begin{array}{c} 1 & a & 1 \\ a+1 & \diagup \quad \diagdown & a+1 \\ & 1 & \\ a & \diagup \quad \diagdown & a \end{array}\right) = \mathcal{F}_n\left(\begin{array}{c} 1 \\ \diagup \quad \diagdown \\ a \end{array}\right) \oplus [n-a-1] \mathcal{F}_n\left(\begin{array}{c} 1 & a-1 & 1 \\ a & \diagup \quad \diagdown & a \\ & 1 & \end{array}\right),$$

$$\mathcal{F}_n\left(\begin{array}{c} a & b \\ \diagup \quad \diagdown \\ a & 1 & b \end{array}\right) = \mathcal{F}_n\left(\begin{array}{c} a & b \\ \diagup \quad \diagdown \\ a & 1 & b \end{array}\right) \oplus [b-a] \mathcal{F}_n\left(\begin{array}{c} \\ a & b \end{array}\right).$$

Now any web reduces to \emptyset using finitely many of these relations. At each step we obtain a projective module.

Projective modules over $\mathbb{Z}[X]$ are free. The relations are those for web evaluation so we get $\text{grk}(\mathcal{F}(\Gamma)) = \langle \Gamma \rangle$. \square

What about the knots?

Recall: we can rewrite the crossings as:

$$\langle \text{X} \rangle = \langle \text{Y} \rangle - q^{-1} \langle \text{II} \rangle, \quad \langle \text{X} \rangle = \langle \text{Y} \rangle - q \langle \text{II} \rangle$$

As in Khovanov homology, we categorify this as complexes, using \mathcal{F}_n as a sort of TQFT:

$$\mathcal{F}_n\left(\text{X}\right) = \underbrace{\mathcal{F}_n\left(\text{Y}\right)}_{\xrightarrow{\text{deg}-1}} \longrightarrow q^{\pm 1} \mathcal{F}_n\left(\text{II}\right) \quad \mathcal{F}_n\left(\text{X}\right) = q \mathcal{F}_n\left(\text{II}\right) \longrightarrow \underbrace{\mathcal{F}_n\left(\text{Y}\right)}_{\xrightarrow{\text{deg}-1}}$$

So the complex associated to an oriented link diagram is

$$C(L) = \bigotimes_{\text{crossings } x} (\text{Complex at } x)$$

Theorem: This is an invariant of oriented links that categorifies the sl_n link polynomial.
(up to overall grading shift) (It's considered up to homotopy)

A generalization for free

In fact we can express any crossing as:

$$\left\langle \begin{array}{c} \nearrow \\ \searrow \\ b \quad a \end{array} \right\rangle = \sum_{k=\max(0, b-a)} (-1)^{k-b} q^{k-b} \left\langle \begin{array}{c} \nearrow \\ \uparrow \downarrow \uparrow \\ b \quad a \end{array} \right\rangle$$

So we can form the more general **Rickard complex**:

$$F_n \left(\begin{array}{c} \nearrow \\ \searrow \\ b \quad a \end{array} \right) = F_n \left(\underbrace{\begin{array}{c} \nearrow \\ \uparrow \downarrow \uparrow \\ b \quad a \end{array}}_{\text{blue}} \right) \rightarrow q^1 F_n \left(\begin{array}{c} \uparrow \downarrow \uparrow \\ b \quad a \\ \uparrow \downarrow \uparrow \\ b-1 \quad b \end{array} \right) \rightarrow q^1 F_n \left(\begin{array}{c} \uparrow \downarrow \uparrow \\ b \quad a \\ \uparrow \downarrow \uparrow \\ b-2 \quad b \end{array} \right) \rightarrow \dots$$

$$\dots \rightarrow q^{1-a} F_n \left(\begin{array}{c} \uparrow \downarrow \uparrow \\ b \quad a \\ \uparrow \downarrow \uparrow \\ b+a \quad b \end{array} \right) \rightarrow q^a F_n \left(\begin{array}{c} \uparrow \downarrow \uparrow \\ b \quad a \\ \uparrow \downarrow \uparrow \\ b-a \quad b \end{array} \right)$$

(Similarly for negative crossings)

$$C_*(L) = \bigotimes_{\text{crossings } x} (\text{Complex at } x)$$

Theorem: This is an invariant of colored oriented links that categorifies the colored sln link polynomial.
 (up to overall grading shift) (It's considered up to homotopy)

5. A cool application

Fact: $H_T^*(\text{Fl}_n) = \frac{\mathbb{Z}[T_i, X_i]}{P(X_i) - P(T_i), P \text{ sym poly}}$

and $H_T^*(\text{Fl}_n(a_1, \dots, a_n)) = H_T^*(\text{Fl}_n)^{S_{a_1} \times \dots \times S_{a_n}}$

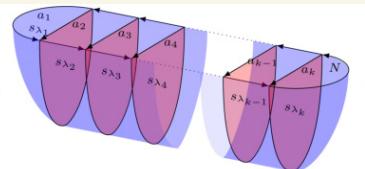
$$\mathbb{C}^{a_1} \subseteq \mathbb{C}^{a_1+a_2} \subseteq \dots \subseteq \mathbb{C}^{a_1+\dots+a_n}$$

Proposition (Robert-Wagner):

$$H_T^*(\text{Fl}_n(a_1, \dots, a_n)) \cong F_n \left(\begin{array}{c} \text{Diagram showing a sequence of rectangles of widths } a_1, a_2, \dots, a_k, N-a_k, N-a_{k+1}, \dots, N-a_n. \\ \text{Arrows indicate horizontal and vertical connections between adjacent rectangles. Labels include } a_1 + a_2, a_1 + a_2 + a_3, \dots, N - a_k, N - a_{k+1}, \dots, N - a_n. \end{array} \right)$$

The isomorphism is given by:

$$\prod_{i=1}^k s_{\lambda_i}(X_{a_i+1}, \dots, X_{a_{i+1}}) \mapsto$$



This can be used to compute the structure

constants for $H_T^*(\text{Fl}_n(a_1, \dots, a_n))$ as a ring:

$$c_{\alpha\beta}^\lambda \stackrel{\text{def}}{=} (-1)^{\sum_{j=2}^k |\widehat{\lambda_j}| + N(N+1)/2} \left(\begin{array}{c} \text{Diagram showing a sequence of rectangles with colored regions labeled } s_{\alpha_2} s_{\beta_2} s_{\alpha_3} s_{\beta_3} s_{\alpha_4} s_{\beta_4}, \dots, s_{\alpha_{k-1}} s_{\beta_{k-1}} s_{\alpha_k}^{-1} s_{\beta_k}^{-1}. \\ \text{Arrows indicate horizontal and vertical connections between adjacent rectangles. Labels include } a_1, a_2, a_3, a_4, a_1+a_2, a_1+a_2+a_3, a_1+a_2+a_3+a_4, N-a_k-a_{k-1}, a_k, N. \end{array} \right).$$

Thank you!

Questions?

