Outline:

- Plethystic subs
- Baez - Moeller - Trimble '21
  Gorsky - Wedrich '19
- Questions for the audience

# 1. Plethystic substitutions

Let $K = \mathbb{Q}(q)$ and $\Lambda_q$ the ring of symmetric functions over $K$.

$$\Lambda_q = K[p_1, p_2, \ldots] \quad \text{power sums: } p_n = x_1^n + x_2^n + \ldots$$

Let $K_b[X]$ = bounded power series in $x_1, x_2, \ldots$

Let $E \in K_b[X]$. Define $p_k[E]$ as the series $E$ where we replace $x_i \mapsto x_i^k$ and $q \mapsto q^k$. This can be extended to a $\mathbb{Q}$-algebra homomorphism

$$\Lambda_q \to K_b[X]$$
$$f \mapsto f[E]$$

This is called the plethystic substitution of $E$ into $f$. Note that if $E \in \Lambda_q$, then $f[E] \in \Lambda_q$.

Example: $h_2\left[\frac{x_1+x_2}{1-q}\right] = \frac{1}{2}\left(p_1^2\left[\frac{x_1+x_2}{1-q}\right] + p_2\left[\frac{x_1+x_2}{1-q}\right]\right) = \frac{1}{2}\left(\frac{x_1+x_2}{1-q}\right)^2 + \frac{1}{2}\frac{x_1^2+x_2^2}{1-q^2} = \frac{h_2(x_1,x_2) + q e_2(x_1,x_2)}{(1-q)^2(1+q)}$

Several important substitutions:

- If $E = X = x_1 + x_2 + \ldots$, then $f \mapsto f[X]$ is the identity.
- If $E = x_1 + \ldots + x_n$, then $f \mapsto f[E]$ is the natural map $\Lambda_q \to K[x_1, \ldots, x_n]^{S_n}$
- If $E = -X$, then $f \mapsto f[-X]$ sends $s_\lambda \leftrightarrow s_{\lambda^T}$
- If $E = Y = y_1 + \ldots$ (a new set of variables) then have $\Delta^+ : \Lambda \to \Lambda \otimes \Lambda$, $f \mapsto f[X+Y]$ $\quad \Delta^\times : \Lambda \to \Lambda \otimes \Lambda$, $f \mapsto f[XY]$

They often simplify notation:

- Let $\Omega[X] = \sum_{n \geq 0} h_n \in \hat{\Lambda}_q$. The "Cauchy identity" reads $\Omega[XY] = \sum_\lambda s_\lambda(X) s_\lambda(Y)$
- Macdonald's $q,t$-Hall inner product is given by $\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$

  Plethystically, this reads $\langle f, g \rangle_{q,t} = \langle f, g\left[\frac{1-q}{1-t}\right] \rangle$

| | Product: | Orth. basis: | Operators: | Eigenvalues of $\nabla^\circ_{-,-}$ |
|---|---|---|---|---|
| | $\langle , \rangle$ = Hall inner product | Schur polys | $\nabla^a(f) = f\left[X + \frac{1}{z}\right]\Omega[-zX]\Big\|_{z^a}$ | $\nabla^\circ(s_\lambda) = 0$ |
| $t=0$ | $\langle , \rangle_t = \langle (-), (-)[X(1-t)] \rangle$ | Hall-Littlewood polys | $\nabla^a_t(f) = f\left[X + \frac{(1-t)}{z}\right]\Omega[-zX]\Big\|_{z^a}$ | $\nabla^\circ_t(P_\lambda) = q^{\lambda_1+1} P_\lambda$ |
| $q=0$ | $\langle , \rangle_{q,t} = \langle (-), (-)\left[X\frac{1-t}{1-q}\right] \rangle$ | Macdonald polys | $\nabla^a_{q,t}(f) = f\left[X + \frac{(1-t)(1-q)}{z}\right]\Omega[-zX]\Big\|_{z^a}$ | $\nabla^\circ_{q,t}(H_\lambda) = (1-t)\sum_{i=1}^\infty q^{\mu_i} t^{i-1} H_\lambda$ |

Keep!

## 2. Schur functors (revisited) Baez-Moeller-Trimble (2021)

| | | $\bigoplus \mathbb{C}[S_n]\text{-mod}$ |
|---|---|---|
| $+$ | $f+g$ | $V \oplus W$ |
| $\cdot$ | $fg$ | $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes_{\mathbb{C}} W)$ |
| $\Delta^+$ | $f[X+Y]$ | $\bigoplus \mathrm{Res}_{S_n \times S_m}^{S_{n+m}}(V \otimes_{\mathbb{C}} W)$ |
| $*$ | $f*g$ | $V \otimes_{\mathbb{C}S_n} W$ |
| $\Delta^\times$ | $f[XY]$ | ? |
| $\circ$ | $f[g]$ | ? |

$(\cdot, *, \Delta^+)$: "Hopf ring" = ring object in the category of corings $\rightarrow$ $\Delta^\times$ and $*$ don't commute

$(\cdot, \Delta^+, \Delta^\times)$: "biring" = coring object in the category of rings $\rightarrow$ Can be defined in terms of plethystic substitutions

↳ Equivalently: $(\cdot, \Delta^+)$ Hopf alg; antipode: $f[-X]$
$(\cdot, \Delta^\times)$ bialg
$\Delta^\times$ codistributes over $\Delta^+$

↳ Equivalently: a ring object in affine schemes

$(\cdot, \Delta^+, \Delta^\times, \circ)$: "plethory" = monoid in the category of birings $\rightarrow$ What they categorify

Another example: $\mathbb{Z}[x]$: $\Delta^+(p(x)) = p(x) + p(y)$, $\Delta^\times(p(x)) = p(xy)$, $p \circ q(x) = p(q(x))$.

Note: By Yoneda, one can define a biring as a ring $B$ such that $\mathrm{Hom}_{\mathrm{Ring}}(B,-)$ has a lift

$$\begin{tikzcd} & \mathrm{Ring} \arrow[d, "\mathrm{forget}"] \\ \mathrm{Ring} \arrow[ur, dashed, "\Phi"] \arrow[r, "\mathrm{Hom}_{\mathrm{Ring}}(B,-)"'] & \mathrm{Set} \end{tikzcd}$$

"functor of points"

For instance, $\Phi = 1_{\mathrm{Ring}}$ is representable by $\mathrm{Hom}(\mathbb{Z}[x], -)$, hence $\mathbb{Z}[x]$ is a biring

Since $1_{\mathrm{Ring}} \circ 1_{\mathrm{Ring}} = 1_{\mathrm{Ring}}$, $\mathbb{Z}[x]$ is a monoid in Biring, i.e. a plethory.

The above can be replicated with rigs instead, aka semirings.

Now let 2-Rig be the 2-category of linear symmetric monoidal additive Karoubian categories, functors and nat. transfs.

Then a 2-birig $B$ is a 2-Rig such that there is a lift

$$\begin{tikzcd} & \text{2-Rig} \arrow[d] \\ \text{2-Rig} \arrow[ur, dashed, "\Phi"] \arrow[r, "\mathrm{Hom}_{\text{2-Rig}}(B,-)"'] & \mathrm{Cat} \end{tikzcd}$$

A 2-plethory is a "pseudomonoid" in 2-Birig, roughly a 2-Birig s.t. $\Phi \circ \Phi \cong \Phi$

**Example:** let Schur be the free 2-Rig on one element $E$, so it has objects $E^{\otimes n}$ as well as

$S^\lambda(E) := (E^{\otimes n}, \pi_\lambda)$. Morphisms: $\mathrm{Hom}(E^{\otimes n}, E^{\otimes m}) = \begin{cases} \mathbb{C}[S_n] & n=m \\ 0 & n\neq m \end{cases}$

**Fact:** Schur represents the identity 2-functor, so Schur is a 2-plethory. It decategorifies to the plethory structure on $\Lambda$.

Description of the 2-plethory structure?

## Abstract Schur functors

Definition: a pseudonatural transformation from $U: 2\text{-Rig} \to \mathrm{Cat}$ to itself.

Write $[V,W]$ = cat of pseudonat. transfs. + modifications, and write $U \times U : 2\text{-Rig} \times 2\text{-Rig} \to \mathrm{Cat} \times \mathrm{Cat}$

**Theorem** (Baez-Moeller-Trimble, 2021). The category Schur is equivalent to $[U,U]$.

Explicitly, the 2-plethory structure on Schur can be carried over to $[U,U]$, and in particular:

$+ \rightsquigarrow \oplus : [U,U] \times [U,U] \to [U,U] \qquad (F,G) \mapsto F \oplus F$

$\cdot \rightsquigarrow \otimes : [U,U] \times [U,U] \to [U,U] \qquad (F,G) \mapsto F \otimes G$

$\circ \rightsquigarrow [U,U] \times [U,U] \to [U,U] \qquad (F,G) \mapsto F \circ G$

$\Delta^+ \rightsquigarrow [U,U] \to [U \times U, U] \qquad F \mapsto F((-)\oplus(-))$

$\Delta^\times \rightsquigarrow [U,U] \to [U \times U, U] \qquad F \mapsto F((-)\otimes(-))$

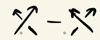
## 3. The annular category   Gorsky - Wedrich (2019)

Let $Schur_q$ be the free graded 2-Rig generated by an object $E$ and a morphism $x : E \to q^{2} E$.

- $Hom(q^k E^{\otimes n}, q^l E^{\otimes n}) = \mathbb{C}[x_1, \dots, x_n] \rtimes \mathbb{C}[S_n]$   "dotted permutations"  $\sigma = X$ , $x = \phi$
- Indecomposables: $q^k S^\lambda(E)$

Rmk: $Schur_q \cong \bigoplus_n (\mathbb{C}[x_1, \dots, x_n] \rtimes \mathbb{C}[S_n])$-gmod

<u>Link invariants</u>

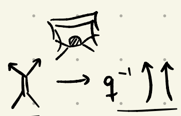
$\beta \longmapsto \hat{\beta} \longmapsto i(\hat{\beta}) \in Sk^+(A) \overset{=}{}$ pos. oriented links on $A$, mod Hecke:  $= (q - q^{-1}) \uparrow\uparrow$

$\longmapsto f_\beta \in \Lambda_q$   map: $x \longmapsto \sum_\lambda Tr(x, V_\lambda) s_\lambda$   Fact (Turaev, '88): $Sk^+(A) \cong \Lambda_q$

Rmk: plethystically, the $sl_N$-link polynomial can be obtained as: $\beta \mapsto \hat{\beta}$ (in annulus) $\mapsto f \longmapsto f\left[\frac{q^N - q^{-N}}{q - q^{-1}}\right]$

the HOMFLY: $f \mapsto f\left[\frac{a - a^{-1}}{q - q^{-1}}\right]$

<u>Link homology</u>

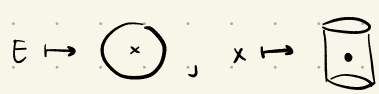
Given  , can send it to  "Rickard complex"

$\beta \mapsto$ tensor of Rickard complexes

Can do this for annular braids, and get a complex in $K^b(\underbrace{Kar(AFoam^+_\infty)}_{\text{Categorifies } Sk^+(A)})$   Category of $gl_\infty$ annular webs and annular foams.

This complex can be thought of as the universal type A categorified invariant of $\beta$

Theorem (Queffelec-Rose, '18): $Kar(AFoam^+_\infty) \cong Schur_q$



Rmk: any graded functor $Schur_q \to \mathcal{C}$ (additive) yields a link invariant, e.g.:

- $E = \mathbb{Z}[a]/a^2$ , $x = a \rightsquigarrow$ Khovanov homology
- $E = \mathbb{Z}[a]/a^2$ , $x = 0 \rightsquigarrow$ Annular Khovanov homology
- $E = \mathbb{Z}[a]/a^n$ , $x = a \rightsquigarrow$ Khovanov Rozansky

Theorem (Gorsky-Wedrich, '19): $\bigoplus_{n \geq 0} Kar(\underbrace{Tr_0(SBim_n)}) \cong Schur_q$

Elias-Lauda: $\cong \mathbb{C}[x_1, \dots, x_n] \rtimes \mathbb{C}[S_n]$

One of their main results is computing the "annular evaluations" of braids such as the half-twist. In $\Lambda_q$, this is:

 $\longmapsto (-1)^n \dfrac{h_n[(q^{-1}-q)X]}{q^{-1}-q}$

To categorify this, they categorify the map $f \longmapsto \dfrac{f[(q^{-1}-q)X]}{q^{-1}-q}$, using "affine extensions"

Affine extension: Given $\mathcal{C}$ graded, let $\mathcal{C}[t]$ =
- Objects: $F[t]$ for $F \in \mathcal{C}$
- Morphisms: $\mathrm{Hom}(F[t], G[t]) = \mathrm{Hom}(F,G) \otimes \mathbb{C}[t]$, where $t$ has $q$-degree 2

Then, consider the functors
$$\pi^* : \mathcal{C} \longrightarrow \mathcal{C}[t], \quad F \longmapsto F[t]$$

$$\pi_* : \mathcal{C}[t] \longrightarrow \mathcal{C}, \quad F[t] \longmapsto F \otimes \mathbb{C}[t] \cong \bigoplus_{i \geq 0} q^{2i} F$$

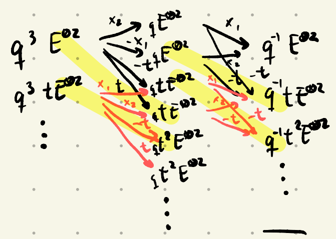
Define a complex $K(E,x) \in \mathrm{Kar}^b(\mathrm{Schur}_q[t])$ by $q\pi^* E \xrightarrow{x-t} \underline{q^{-1}\pi^* E}$. Then we have

$$f \longmapsto f[X(q^{-1}-q)] \longmapsto \frac{f[X(q^{-1}-q)]}{q^{-1}-q}$$

$$F \longmapsto F(K(E,x)) \longmapsto \pi_*(q\, F(K(E,x)))$$

Examples: if $F = E$, then
$$\pi_*(q\,K(E,x)) = \begin{tikzcd} q^2E \arrow[r, "x"] \arrow[dr, "-t"] & E \\ tq^2E \arrow[r, "x"] \arrow[dr, "-t"] & tE \\ t^2q^2E \arrow[r, "x"] \arrow[dr, "-t"] & t^2E \\ & \end{tikzcd} \overset{\text{Gauss elim}}{\simeq} \underline{E}$$

if $F = E^{\otimes 2}$, then
$$K(E,x)^{\otimes 2} = q^{-2}E^{\otimes 2}[t] \xrightarrow{\begin{pmatrix} x_2-t \\ -x_1+t \end{pmatrix}} \begin{matrix} E^{\otimes 2}[t] \\ \oplus \\ E^{\otimes 2}[t] \end{matrix} \xrightarrow{(x_1-t,\ x_2-t)} q^2 E^{\otimes 2}[t]$$

$\pi_*(q\,K(E,x)^{\otimes 2}) =$  $\simeq q\,E^{\otimes 2} \xrightarrow{x_1-x_2} \underline{q^{-1}E^{\otimes 2}}$

Compare with: $\dfrac{p_1^2[X(q-q^{-1})]}{q-q^{-1}} = (q-q^{-1})\, p_1^2[X]$.

## 4. Questions

1. Does $\mathrm{Hom}(F, G[X(1-q^2)])$ categorify $\langle , \rangle_q$?

2. Can one express Hall-Littlewood polys in this context?

Answer (Eugene): 1. Yes!

2. Unclear. A bit tautologically:

Fact: HL poly $P_\mu = \underset{\text{graded Frobenius character}}{q\mathrm{ch}}\, H^*(B_\mu)$, and $H^*(B_\mu) = \mathbb{C}[x_1, \dots, x_n]/I_\mu$

Take a free res of $I_\mu$, and replace $\mathbb{C}[x_1,\dots,x_n]$ by $E^{\otimes n}$. The resulting complex in $K^b(\mathrm{Schur}_q)$ categorifies $P_\mu$

Example: $H^*(B_{\square\square}) = \mathbb{C}[x,y]/(x+y, y^2)$, so take $\mathbb{C}[x,y] \xrightarrow{\binom{-y^2}{x+y}} \mathbb{C}[x,y]^{\oplus 2} \xrightarrow{(x+y,\, y^2)} \mathbb{C}[x,y] \rightsquigarrow E^{\otimes 2} \xrightarrow{\binom{-y^2}{x+y}} E^{\otimes 2} \xrightarrow{(x+y,\, y^2)} E^{\otimes 2}$

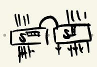
Question 3 (Eugene): can one reinterpret those in terms of SSBims? Resolutions don't need to be free...

↳ Or a complex in $K^b(\mathrm{AFoam}^+_\infty)$?

Question 4: can one characterize HL complexes in terms of $\mathrm{Hom}_{K^b}$? Can one prove their positivity?

Finally, Khovanov introduced a diagrammatic category $\mathcal{H}$ that fits in the following picture:

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \bigoplus \mathbb{C}S_n\text{-mod} \\ \downarrow & & \downarrow K_0 \\ \mathfrak{h} & \xrightarrow{\text{Fock}} & \Lambda \end{array}$$

This has objects $\uparrow\downarrow\uparrow\downarrow\cdots\downarrow$ and it's Karoubian, so one has Schur functors 

Cautis-Sussan introduced certain complexes $\cdots \to \bigoplus_{n=m+a} S^{\lambda}(\uparrow^{\otimes n}) \otimes S^{\mu}(\downarrow^{\otimes m}) \longrightarrow \bigoplus_{m=m+a-1} S^{\lambda'}(\uparrow^{\otimes n}) \otimes S^{\mu'}(\downarrow^{\otimes m}) \to \cdots$

that were shown by González to categorify the Boson-Fermion correspondence, roughly an action of the Clifford algebra on $\Lambda$.

These are actually categorified Bernstein operators!

Question 5: is there a (graded?) Heisenberg category that acts on $\mathrm{Schur}_q$ s.t. one can define categorified Jing operators? The objects might be HL complexes. The case of $\nabla_t$ would already be interesting since $H_\mu$ is an eigenfunction of $\nabla_t^\circ$ with eigenvalue $q^{\mu_1+1}$.

Question 6: can any of this be done for Macdonald polynomials? (Explain Garsia-Haiman modules if there's time).