HW4 Álvaro

The octonionic projective plane $\mathbb{O}P^2$ and map $S^{15}\to S^8$ with Hopf invariant 1.

Fomenko-Fuchs isn't too detailed so I'll be following in addition a note on this subject by Malte Lackmann.

a. Construction of the octonions: the octonions are a particular instance of a general construction. If $(A, \|\cdot\|, *)$ is some normed $\mathbb{R}$-algebra with a conjugation, then A^2 can be made into a normed $\mathbb{R}$-algebra with a conjugation by setting

- $(a,b)\cdot(c,d) = (ac - d^*b,\ da + bc^*)$
- $\|(a,b)\| = \sqrt{\|a\|^2+\|b\|^2}$
- $(a,b)^* = (a^*, -b)$

Examples: $A=\mathbb{R} \Rightarrow A^2\cong\mathbb{C}$. $A=\mathbb{C}\Rightarrow A^2\cong\mathbb{H}$. $A=\mathbb{H}\to A^2\cong\mathbb{O}$

b. We use a slightly different definition of $\mathbb{O}P^2$ (which cannot be naively defined as a quotient of $\mathbb{O}^3\setminus 0$ due to the absence of associativity in $\mathbb{O}$)
Define $T=\{(x,y,z)\in\mathbb{O}^3 : \|x\|^2+\|y\|^2+\|z\|^2=1$ and the algebra generated by x,y,z is associative$\}$.
Define an equivalence relation $(a,b,c)\sim(x,y,z) \Leftrightarrow$ all of the following equations hold: $aa^*=xx^*$, $bb^*=yy^*$, $cc^*=zz^*$, $ab^*=xy^*$, $ac^*=xz^*$, $bc^*=yz^*$

Then $\mathbb{O}P^2 := T/\sim$

c. $\mathbb{O}P^2$ is a 16-dim'l manifold. We define charts as follows. Pick a triple $a,b,c\in\mathbb{R}$ and define $\ell_{(a,b,c)}:\mathbb{O}^3\to\mathbb{O}$, $(x,y,z)\mapsto ax+by+cz$.
Then it's easily checked that if $(x,y,z)\sim(x',y',z')$, we have $\ell_{(a,b,c)}(x,y,z)=0 \Leftrightarrow \ell_{(a,b,c)}(x',y',z')$.
It follows that we may define an open set $U_{(a,b,c)} = \{[(x,y,z)]\in\mathbb{O}P^2 : \ell_{(a,b,c)}(x,y,z)\neq 0\}$

Now the chart is
$$\varphi_{(a,b,c)}: U_{(a,b,c)} \to \mathbb{O}^2,\quad [x,y,z]\mapsto\left(\frac{x\ell^*}{\|\ell\|^2}, \frac{y\ell^*}{\|\ell\|^2}\right), \text{ where } \ell=\ell_{(a,b,c)}(x,y,z)$$

We omit the expression for the inverse map $\psi_{(a,b,c)}:\mathbb{O}^2\to U_{(a,b,c)}$. The fact that $\psi_{(a,b,c)}$ is well defined relies on the fact that in $\mathbb{O}$, any subalgebra generated by two elements is associative (this is called alternativity). This is not true for 3 or more elements, so an analogous map $\mathbb{O}^n\to U_{(a_1,\dots,a_{n+1})}$ would not be well defined for $n\geq 3$.
It is easy to see that $\mathbb{O}P^2$ is Hausdorff. Since T is compact, we conclude that $\mathbb{O}P^2$ is a closed 16-dim'l manifold.

d. CW structure. Entirely analogously to the real, complex and quaternionic cases, we have an embedding $\mathbb{O}P^1\hookrightarrow\mathbb{O}P^2$, $[x,y]\mapsto[x,y,0]$ and $\mathbb{O}P^1\cong S^8$. Now we have a normed space isomorphism $\mathbb{O}\cong\mathbb{R}^8$, so we can consider
$$g:\mathbb{R}^8\times\mathbb{R}^8\cong\mathbb{O}^8\times\mathbb{O}^8\to\mathbb{O}P^1\cong S^8,\quad (x,y)\mapsto[x,y]$$
Define $f=g|_{S^{15}}$. We may extend f to $\tilde{f}:D^{16}\to\mathbb{O}P^2$, $(x,y)\mapsto[x,y,\sqrt{1-\|x\|^2-\|y\|^2}]$ and one checks that this induces $\mathbb{O}P^1\cup_f D^{16}\underset{\text{homeo}}{\cong}\mathbb{O}P^2$.

In particular, $H^*(\mathbb{O}P^2,A)=\begin{cases}A & *=0,8,16\\ 0 & \text{o/w}\end{cases}$

e. Hopf invariant. Let τ generate $H^8(\mathbb{O}P^2)$ and σ generate $H^{16}(\mathbb{O}P^2)$. We show that $\tau^2=\sigma$ up to sign.
It is clear that $\mathbb{O}P^2$ is simply-connected so in particular it is oriented. Now
$$H^8(\mathbb{O}P^2)\underset{\text{UCT}}{\cong}\mathrm{Hom}(H_8(\mathbb{O}P^2),\mathbb{Z})\underset{\text{PD}}{\cong}\mathrm{Hom}(H^8(\mathbb{O}P^2),\mathbb{Z})$$
$$\alpha\mapsto(\sigma\mapsto\alpha(\sigma))\mapsto(\beta\mapsto\langle\alpha,[\mathbb{O}P^2]\cap\beta\rangle=\langle\alpha\cup\beta,[\mathbb{O}P^2]\rangle)$$
In particular τ gets sent to a generator of $\mathrm{Hom}(H^8(\mathbb{O}P^2),\mathbb{Z})\cong\mathbb{Z}$, so $\langle\tau\cup\tau,[\mathbb{O}P^2]\rangle=\pm1$
Therefore since $\tau^2=h(f)\sigma$, $\pm1=\langle\tau^2,[\mathbb{O}P^2]\rangle = h(f)\underbrace{\langle\sigma,[\mathbb{O}P^2]\rangle}_{\in\mathbb{Z}}$, so $h(f)=\pm1$, as desired. (Adjusting orientations, we get the sign).