1 Hindman's Theorem

We illustrate an approach to topological dynamics via ultrafilters, using Hindman's Theorem as an example. The statement had been conjectured in 1968 [1] and popularised in 1972 by its appearance in [2] and Erdős' interest in the problem. In 1974, Hindman proved the result using a complicated combinatorial argument.

Definition 1.1. Given $B \subset \mathbb{N}$, the set of finite sums of distinct elements of B is

$$FS(B) = \left\{ \sum_{b \in C} b : C \subset B,\ C \text{ is finite} \right\}.$$

Theorem 1.1 (Hindman, 1974)**.** *For any finite colouring of $\mathbb{N}$, there exists an infinite set $B \subset \mathbb{N}$ such that $FS(B)$ is monochromatic.*

This is a substantial strengthening of Schur's Theorem. We will also prove as a corollary Folkman's Theorem, the generalisation of Schur's Theorem for sets of any finite size.

This proof follows [3], but was originally due to Glazer and Galvin (1975). The simplicity of the arguments that resulted from the algebraic translation of topological dynamics raised great interest; similar ideas would later be used in order to prove Hales-Jewett, Ramsey and a number of other Ramsey-type theorems.

The translation is as follows. Instead of the usual colouring shift map $T : X \to X$ (see [4]), we consider the underlying semigroup $\mathbb{N}$ that acts on X by $n * c = T^n c$. We then embed $\mathbb{N}$ (with the discrete topology) into a compact Hausdorff space $\beta\mathbb{N}$, such that the action of $\mathbb{N}$ extends to an action of $\beta\mathbb{N}$. In this case we will have $\{T^n c : n \in \beta\mathbb{N}\} = \overline{\{T^n c : n \in \mathbb{N}\}}$. The space $\beta\mathbb{N}$ is called the Stone-Čech compactification of $\mathbb{N}$. To this end, the compactification should have the property that $\mathbb{N}$ is a dense subset.

We will seek an element $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. This implies the existence of a particular kind of recurrent point in the space of colourings. Using this, we prove Hindman's Theorem.

We start by stating the definition of an ultrafilter.

Definition 1.2. Let X be a set. An *ultrafilter on X* is a collection $\mathcal{U}$ of subsets of X such that:

(i) $\emptyset \notin \mathcal{U}$;
(ii) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
(iii) if $A \supset B$ and $B \in \mathcal{U}$, then $A \in \mathcal{U}$;
(iv) if $A \subset X$, either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$.

A family of subsets satisfying only (i), (ii) and (iii) is called a filter.

Lemma 1.1. *Given a family of sets $\mathcal{A} = \{A_i : i \in I\}$ such that no finite intersection of them is empty, there exists an ultrafilter $\mathcal{U}$ containing the family $\mathcal{A}$.*

Proof. We can 'complete' the collection $\mathcal{A}$ adding all finite intersections in $\mathcal{A}$ and supersets of these sets. This collection is clearly a filter.

We use Zorn's lemma. The collection of filters containing $\mathcal{A}$ is partially ordered under set inclusion. Since every chain has an upper bound, namely the union of all the filters in the chain, by Zorn's lemma there exists some maximal filter $\mathcal{U}$.

Now if $A \notin \mathcal{U}$, by filter maximality, adding all supersets of A to $\mathcal{U}$ would fail (ii). So there is $B \in \mathcal{U}$ satisfying $A \cap B = \emptyset$. Now $B \subset A^C$, so $A^C \in \mathcal{U}$. So $\mathcal{U}$ is an ultrafilter. □

For our purposes, we may now restrict our attention to the ultrafilters on $\mathbb{N}$.

Definition 1.3. The *Stone-Čech compactification of* $\mathbb{N}$ is

$$\beta\mathbb{N} = \{\mathcal{U} \subset \mathcal{P}(\mathbb{N}) : \mathcal{U} \text{ is an ultrafilter on } \mathbb{N}\}.$$

Equip $\beta\mathbb{N}$ with the topology generated by open sets of the form $\langle A\rangle = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}$, where $A \subset \mathbb{N}$. Note that the elements of the basis are closed under intersections, so the topology is well defined.

Additionally, whenever $A \subset \mathbb{N}$, we have $\beta\mathbb{N} \setminus \langle A\rangle = \langle \mathbb{N} \setminus A\rangle$, and so every set in the basis is clopen (both open and closed).

Definition 1.4. An ultrafilter of the form $\mathcal{U}_n = \{A \subset \mathbb{N} : n \in A\}$ for some $n \in \mathbb{N}$ is *principal*.

An ultrafilter is principal if and only if it contains a finite set. Indeed, if $\mathcal{U}$ contains $A = \{a_1, ..., a_m\}$, we note that $\bigcap_i A \setminus \{a_i\} = \emptyset \notin \mathcal{U}$. So $\mathcal{U}$ cannot contain all the sets $\{a_1\}^C, ..., \{a_m\}^C$, and thus it must contain some singleton $\{a_i\}$. It is then clear that $\mathcal{U}$ is principal.

The embedding $\mathbb{N} \to \beta\mathbb{N}$ is given by $n \mapsto \mathcal{U}_n$. It is easy to see that every open set in $\beta\mathbb{N}$ contains some $\mathcal{U}_n$, and so the image of $\mathbb{N}$ is dense in $\beta\mathbb{N}$.

Proposition 1.2. *The space $\beta\mathbb{N}$ is compact.*

Proof. We use the following formulation of compactness: assume that $\{\langle A_i\rangle\}_{i\in I}$ is a family of complements of basic open sets whose finite intersections are nonempty. Our aim will be to show that

$$\bigcap_{i\in I} \langle A_i\rangle \neq \emptyset. \tag{1}$$

It is clear that $\langle A_{i_1}\rangle \cap \cdots \cap \langle A_{i_m}\rangle = \langle A_{i_1} \cap \cdots \cap A_{i_m}\rangle$ and that $\langle C\rangle = \emptyset$ iff $C = \emptyset$. Now as the $\langle A_i\rangle$ have nonempty finite intersections, so do the A_i. Hence by Lemma 1.1 we may take an ultrafilter $\mathcal{U}$ such that $A_i \in \mathcal{U}$ for all $i \in I$. Clearly this element lies in the intersection (1). □

Proposition 1.3. *The space $\beta\mathbb{N}$ is Hausdorff.*

Proof. Take $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ with $\mathcal{U} \neq \mathcal{V}$. Pick $A \subset \mathbb{N}$ such that $A \in \mathcal{U}$ but $A \notin \mathcal{V}$. Then $A^C \in \mathcal{V}$ by (iv) and so $\mathcal{U} \in \langle A \rangle$, $\mathcal{V} \in \langle A^C \rangle$ and, since ultrafilters cannot include disjoint sets, $\langle A \rangle \cap \langle A^C \rangle = \emptyset$. □

We now show that the notion of addition in $\mathbb{N}$ extends to an addition in $\beta\mathbb{N}$. For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$, we define $A - n = \{a - n : a \in A\}$.

Definition 1.5. Given two ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, the sum of $\mathcal{U}$ and $\mathcal{V}$ is given by

$$\mathcal{U} \oplus \mathcal{V} = \{A \subset \mathbb{N} : \{n \in \mathbb{N} : A - n \in \mathcal{U}\} \in \mathcal{V}\}.$$

One checks that the sum of two ultrafilters is an ultrafilter, and that this operation is associative. To gain some intuition on this addition, we prove that it actually extends the usual addition on the natural numbers.

Proposition 1.4. *For any $a, b \in \mathbb{N}$, the principal ultrafilters $\mathcal{U}_a, \mathcal{U}_b \in \beta\mathbb{N}$ satisfy*

$$\mathcal{U}_a \oplus \mathcal{U}_b = \mathcal{U}_{a+b}.$$

Proof. The elements in $\mathcal{U}_a \oplus \mathcal{U}_b$ are precisely the subsets $A \subset \mathbb{N}$ satisfying

$$\{n \in \mathbb{N} : A - n \in \mathcal{U}_a\} \in \mathcal{U}_b.$$

Recall that any particular subset of natural numbers lies in the ultrafilter $\mathcal{U}_b$ if and only if it contains the element b. Therefore these are the subsets satisfying

$$b \in \{n \in \mathbb{N} : A - n \in \mathcal{U}_a\}.$$

In other words, these are the subsets such that $A - b \in \mathcal{U}_a$. But this condition is the same as $a \in A - b$, which is equivalent to $a + b \in A$. Therefore the elements in $\mathcal{U}_a \oplus \mathcal{U}_b$ are precisely the subsets containing $a + b$, i.e. the elements in $\mathcal{U}_{a+b}$.

□

It is perhaps surprising that this addition is not commutative. In fact, the sum commutes if and only if at least one of the elements is a natural number [4]. We now prove the existence of an idempotent, the key element for the proof of Theorem 1.1. We will need the following lemma.

Lemma 1.5. *Let $\mathcal{U} \in \beta\mathbb{N}$. The map $\Psi_{\mathcal{U}} : \beta\mathbb{N} \to \beta\mathbb{N}$ given by $\Psi_{\mathcal{U}}(\mathcal{V}) = \mathcal{U} \oplus \mathcal{V}$ is continuous.*

Proof. We consider the preimage of an element of the basis $\langle A \rangle$. We have $\Psi_{\mathcal{U}}^{-1}(\langle A \rangle) = \{\mathcal{V} \in \beta\mathbb{N} : A \in \mathcal{U} \oplus \mathcal{V}\} = \{\mathcal{V} \in \beta\mathbb{N} : \{n : A - n \in \mathcal{U}\} \in \mathcal{V}\} = \langle \{n : A - n \in \mathcal{U}\} \rangle$. This is open. $\square$

Theorem 1.2 (Ellis-Numakura for $\beta\mathbb{N}$, 1958)**.** *The space $\beta\mathbb{N}$ contains some idempotent, that is, there exists $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$.*

Proof. For $S, T \subset \beta\mathbb{N}$, define $S \oplus T = \{\mathcal{U} \oplus \mathcal{V} : \mathcal{U} \in S,\ \mathcal{V} \in T\}$. Consider

$$\mathbb{P} = \{S \subset \beta\mathbb{N} : S \text{ closed},\ S \neq \emptyset,\ S \oplus S \subset S\}.$$

This is partially ordered under inclusion. We know $\beta\mathbb{N}$ is compact by Proposition 1.2, so all elements of $\mathbb{P}$ are compact. Hence, by Cantor's intersection theorem, every descending chain of such sets has nonempty intersection, an element of $\mathbb{P}$. Thus, by Zorn's lemma, there exists some minimal nonempty set $K \in \mathbb{P}$. Take some $\mathcal{U} \in K$. We will show that $\mathcal{U}$ is idempotent.

First, we show that $K = \{\mathcal{U}\} \oplus K$. Let $K' = \{\mathcal{U}\} \oplus K$. Notice that $\mathcal{U} \oplus \mathcal{U} \in K'$ so $K' \neq \emptyset$. We also have that

$$K' \oplus K' = \{\mathcal{U}\} \oplus K \oplus \{\mathcal{U}\} \oplus K \subset \{\mathcal{U}\} \oplus K \oplus K \oplus K \subset \{\mathcal{U}\} \oplus K = K'.$$

Further, K' is the image of the compact set K under the continuous map from Lemma 1.5 to a Hausdorff space (by Proposition 1.3). Hence K' is closed and so we conclude $K' \in \mathbb{P}$. Since $K' = \{\mathcal{U}\} \oplus K \subset K \oplus K \subset K$ and K is minimal, necessarily $K = K'$.

Now we consider

$$K'' = \{\mathcal{V} \in K : \mathcal{U} \oplus \mathcal{V} = \mathcal{U}\}.$$

We have just shown that K'' is nonempty, since $\{\mathcal{U}\} \oplus K = K \ni \mathcal{U}$. K'' is also closed because it is the intersection of the closed sets $\{\mathcal{V} \in \beta\mathbb{N} : \mathcal{U} \oplus \mathcal{V} = \mathcal{U}\}$ and K, the former being the preimage of the closed set $\{\mathcal{U}\} = \bigcap_{A \in \mathcal{U}} \langle A \rangle$ under the continuous map from Lemma 1.5.

Finally, to show $K'' \oplus K'' = K''$, take $\mathcal{V}, \mathcal{W} \in K''$ arbitrary. Then

$$\mathcal{U} \oplus (\mathcal{V} \oplus \mathcal{W}) = (\mathcal{U} \oplus \mathcal{V}) \oplus \mathcal{W} = \mathcal{U} \oplus \mathcal{W} = \mathcal{U}.$$

So $K'' \in \mathbb{P}$, and again by minimality, $K'' = K$. Therefore $\mathcal{U} \in K''$ and so $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. $\square$

Finally, we are ready to prove Theorem 1.1. The following part of the proof is a simplification due to Strauss [5, 6].

Proof of Theorem 1.1. Suppose $\mathbb{N} = C_1 \cup \cdots \cup C_k$ is the partition into the colour classes.

Pick an idempotent $\mathcal{U}$ in $\beta\mathbb{N}$. It contains some $A = C_i$, as otherwise it would contain $C_1^C \cap \cdots \cap C_k^C = \emptyset$. We construct a sequence $B = \{n_1, n_2, \ldots\}$ such that $FS(B) \subset A$.

Let $A^* = \{n \in A : A - n \in \mathcal{U}\}$. As $\mathcal{U}$ is idempotent, $A^* \in \mathcal{U}$. We note that this is true for any set $A \in \mathcal{U}$. Choose some $n_1 \in A^*$; we note that trivially $FS(\{n_1\}) \subset A^*$. We proceed inductively. Assume that we have found distinct $n_1, ..., n_m$ with $F = FS(\{n_1, \ldots, n_m\}) \subset A^*$.

Claim: For each $\nu \in F$, $A^* - \nu \in \mathcal{U}$.

By definition, if $\nu \in A^*$, we have $A - \nu \in \mathcal{U}$. Hence the set $(A - \nu)^* = \{n \in A - \nu : (A - \nu) - n \in \mathcal{U}\}$ is an element of $\mathcal{U}$. Now

$$\begin{aligned}(A - \nu)^* &= \{n \in A - \nu : A - \nu - n \in \mathcal{U}\} \\ &= \{n' \in A : A - n' \in \mathcal{U}\} - \nu \\ &= A^* - \nu,\end{aligned}$$

where we have set $n = n' - \nu$. Thus $A^* - \nu \in \mathcal{U}$. This proves the claim.

Note that F is finite. Analogously with the proof of Lemma **??**, pick

$$n_{m+1} \in A^* \cap \bigcap_{\nu \in F} (A^* - \nu) \in \mathcal{U}$$

Clearly $FS(\{n_1, \ldots, n_{m+1}\}) \subset A^*$. Notice at this point that since no element in $\mathbb{N}$ is idempotent, the sets in $\mathcal{U}$ must all be infinite. Therefore, we may take n_{m+1} strictly greater than $n_1, \ldots, n_m$. Letting $B = \{n_1, n_2, n_3, \ldots\}$ yields $FS(B) \subset A^* \subset A$, as required. $\square$

As a corollary of Theorem 1.1, we immediately obtain Folkman's Theorem:

Theorem 1.3 (Folkman, 1968)**.** *Given $r \in \mathbb{N}$, in any finite colouring of $\mathbb{N}$ there exists a finite subset $S \subset \mathbb{N}$ of size r such that $FS(S)$ is monochromatic.*

References

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