Periodicity of Clifford algebras

In lectures we defined, for a quadratic form q on a real vector space V, $C(V,q) = T(V) / \langle x\otimes x - q(x)\cdot 1 : x\in V\rangle$, where $T(V)$ = tensor algebra of V.

Define $C_k := C(\mathbb{R}^k, -\sum_{i=1}^{k} x_i^2) \cong \mathbb{R}\langle e_1,\dots,e_k\rangle / e_i^2=-1,\ e_ie_j=-e_je_i$

The periodicity proposition states:

(a)
$C_1\cong\mathbb{C}$ $\quad C_5\cong\mathbb{C}(4)$
$C_2\cong\mathbb{H}$ $\quad C_6\cong\mathbb{R}(8)$
$C_3\cong\mathbb{H}^2$ $\quad C_7\cong\mathbb{R}(8)\oplus\mathbb{R}(8)$
$C_4\cong\mathbb{H}(2)$ $\quad C_8\cong\mathbb{R}(16)$

and (b) $C_{k+8}\cong C_k\otimes C_8\cong C_k\otimes\mathbb{R}(16)$, so that if $C_k\cong F_1(m)\oplus\dots\oplus F_t(m)$, then $C_{k+8}\cong F_1(16m)\oplus\dots\oplus F_t(16m)$ (hence "periodicity")

We will prove these by introducing $C_k' = \mathbb{R}\langle e_1',\dots,e_k'\rangle / e_i'^2=+1,\ e_i'e_j'=-e_j'e_i'$

lemma: $C_k\otimes_{\mathbb{R}} C_2'\cong C_{k+2}'$
$C_k'\otimes_{\mathbb{R}} C_2\cong C_{k+2}$

Proof: define a linear map $\psi: \mathrm{Span}_{\mathbb{R}}\{e_i'\}\subset C_{k+2}' \to C_k\otimes C_2'$

$$e_i'\mapsto\begin{cases}1\otimes e_i' & i=1,2\\ e_{i-2}\otimes e_1'e_2' & i\geq 3\end{cases}$$

Then
$$\psi(e_i')^2=\begin{cases}1\otimes e_i'^2 = 1\\ e_{i-2}^2\otimes e_1'e_2'e_1'e_2' = -1\otimes(-e_1'^2e_2'^2) = 1\end{cases}$$

so by the universal property of C_{k+2}', we have

$$\begin{tikzcd} C_{k+2}' \arrow[dr, dashed, "\tilde\psi"] & \\ \mathbb{R}^{k+2} \arrow[u] \arrow[r, "\psi"'] & C_k\otimes C_2' \end{tikzcd}$$

The map $\tilde\psi$ is clearly injective so it is an isomorphism by dimension. The other case is analogous. □

Now we have $C_1=\mathbb{C}$, $C_2=\mathbb{H}$, $C_1'=\mathbb{R}[x]/(x^2-1)=\mathbb{R}^2$, $C_2'=\mathbb{R}\langle x,y\rangle/\langle xy=-yx,\ x^2=y^2=1\rangle \xrightarrow{\sim} \mathbb{R}(2)$

$1\mapsto\begin{pmatrix}1&\\&1\end{pmatrix}$
$x\mapsto\begin{pmatrix}&1\\1&\end{pmatrix}$
$y\mapsto\begin{pmatrix}-1&\\&1\end{pmatrix}$
$xy\mapsto\begin{pmatrix}&1\\-1&\end{pmatrix}$

Using the lemma,
$C_3=\mathbb{H}\otimes_{\mathbb{R}}\mathbb{R}^2=\mathbb{H}^2$
$C_3'=\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{C}(2)$
$C_4=\mathbb{R}(2)\otimes_{\mathbb{R}}\mathbb{H}=\mathbb{H}(2)$
$C_4'=\mathbb{H}\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{H}(2)$
$C_5=\mathbb{C}(2)\otimes_{\mathbb{R}}\mathbb{H}=\mathbb{C}(2)(2)=\mathbb{C}(4)$ (since $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}\xrightarrow{\sim}\mathbb{C}(2)$ via $1\otimes_{\mathbb{R}} i\mapsto\begin{pmatrix}i&\\&i\end{pmatrix}$, $1\otimes_{\mathbb{R}} j\mapsto\begin{pmatrix}&-1\\1&\end{pmatrix}$)
$C_5'=\mathbb{H}^2\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{H}^2(2)$
$C_6=\mathbb{H}(2)\otimes_{\mathbb{R}}\mathbb{H}=\mathbb{R}(8)$ (since $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}\xrightarrow{\sim}\mathrm{End}_{\mathbb{R}}(\mathbb{H})=\mathbb{R}(4)$, $a\otimes_{\mathbb{R}} b\mapsto(c\mapsto acb)$)
$C_6'=\mathbb{H}(2)\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{H}(4)$
$C_7=\mathbb{H}^2(2)\otimes_{\mathbb{R}}\mathbb{H}=\mathbb{R}^2(8)$
$C_7'=\mathbb{C}(4)\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{C}(8)$
$C_8=\mathbb{H}(4)\otimes_{\mathbb{R}}\mathbb{H}=\mathbb{R}(16)$
$C_8'=\mathbb{R}(8)\otimes_{\mathbb{R}}\mathbb{R}(2)=\mathbb{R}(16)$

This proves (a).

For (b), note that $C_{k+4}\cong C_{k+2}'\otimes C_2\cong C_k\otimes\underbrace{C_2'\otimes C_2}_{\mathbb{H}(2)}=C_k\otimes C_4$ so in particular $C_{k+8}\cong C_k\otimes\underbrace{C_4\otimes C_4}_{\mathbb{R}(16)}=C_k\otimes C_8$.