Lecture 8:

Solutions to the in-class exercises

1. Compute the image of the transformation with matrix $\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, by finding a basis for it.

<u>Solution 1</u>: If we put the matrix into rref, we get $\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$. Since the linear combinations of the columns of A are exactly those of its rref, we see that the columns $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ are a basis. The column $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is $\frac{1}{2} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Solution 2: We saw that $\dim(\mathrm{Ker}(A)) = 1$, so
$$\dim(\mathrm{Im}(A)) = \dim(\mathbb{R}^3) - \dim(\mathrm{Ker}(A)) = 3 - 1 = 2.$$

So it's enough to find 2 linearly independent vectors in $\mathrm{Im}(A)$.

3. Solution 1: Note that $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x+y+z = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ -x-y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

Now take some pair of vectors such as $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Claim: these form a basis.

- Linear independence: if $\lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $\begin{pmatrix} \lambda \\ \mu \\ -\lambda-\mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda = \mu = 0$ ✓.
- They span S: a vector in S is of the form $\begin{pmatrix} \lambda \\ \mu \\ -\lambda-\mu \end{pmatrix}$ and this is $\lambda \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Solution 2: Note that S is the kernel of the transformation
$$T: \mathbb{R}^3 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x+y+z$$
(matrix of the transformation: $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$)

It is easy to see that $\mathrm{Im}(T) = \mathbb{R}$.

Rank-nullity: $\dim(\mathrm{Ker}(T)) + \underbrace{\dim(\mathrm{Im}(T))}_{= \dim(\mathbb{R}) = 1} = \underbrace{\dim(\mathbb{R}^3)}_{= 3}$

So $\dim(\mathrm{Ker}(T)) = 2$. Therefore it suffices to find two linearly independent vectors in S.

For instance, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Big picture

- Solving linear of equations: Gaussian elimination.

- Vectors

$v_1, \dots, v_m \in \mathbb{R}^n$ are linearly independent $\iff$ The only linear combination $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$ is $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$.

$$\iff \operatorname{rank}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix} = n.$$

$v_1, \dots, v_m \in \mathbb{R}^n$ are a spanning set $\iff$ Every $w \in \mathbb{R}^n$ can be written as $w = \lambda_1 v_1 + \dots + \lambda_m v_m$.

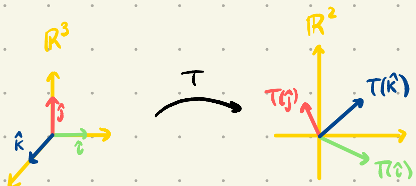
$$\iff \operatorname{rank}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix} = n.$$

- Linear transformations: $T: \mathbb{R}^n \to \mathbb{R}^m$ given by matrices:

$$\left.\overbrace{\left(\boxed{T\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}\ \boxed{T\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}}\ \cdots\ \boxed{T\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}}\right)}^{n}\right\}m$$

$$\begin{cases} T(v+w) = T(v) + T(w) \\ T(\lambda v) = \lambda T(v) \end{cases}$$

Picture:



matrix: $\left.\overbrace{\begin{pmatrix} | & | & | \\ T(\hat{\imath}) & T(\hat{\jmath}) & T(\hat{k}) \\ | & | & | \end{pmatrix}}^{3}\right\}2$

- The Kernel and image of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

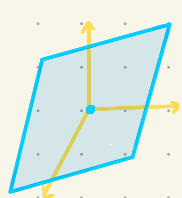
$\operatorname{Ker}(T) = \{v \in \mathbb{R}^n : T(v) = 0\}$ Larger $\operatorname{Ker}(T) \Rightarrow$ "less injective" T is.

$\operatorname{Im}(T) = \{T(v) \in \mathbb{R}^m : v \in \mathbb{R}^n\}$ Larger $\operatorname{Im}(T) \Rightarrow$ "More surjective" T is.

- Subspaces

$$\begin{cases} s_1, s_2 \in S \Rightarrow s_1 + s_2 \in S \\ s \in S,\ \lambda \in \mathbb{R} \Rightarrow \lambda s \in S \end{cases}$$

Picture:



2-dimensional subspace of $\mathbb{R}^3$

Subspaces have bases: $v_1, \dots, v_m \begin{cases} \text{l.i.} \\ \operatorname{Span}(v_1, \dots, v_m) = S \end{cases}$ $\iff$ maximal lin. indep. in S $\iff$ minimal spanning set for S

Number of vectors in any basis is $\dim(S)$.

To find a basis knowing $\dim(S) = d$. It suffices to find $\begin{cases} s \text{ l.i. vectors in } S \Rightarrow \text{they automatically span } S \\ s \text{ vectors that span } S \Rightarrow \text{they're automatically l.i.} \end{cases}$

- Bases of Ker(T) and Im(T)

  Ker: solve $\left(A \,\middle|\, \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix}\right) \rightsquigarrow \left(\mathrm{rref}(A) \,\middle|\, \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix}\right) \rightsquigarrow$ one vector for each free variable (i.e. for each non-pivot column)

  $\Rightarrow$ Basis ✓

  Im: $A \rightsquigarrow \mathrm{rref}(A) \rightsquigarrow$ Basis: columns i of A such that column i of A has a pivot.

- Rank-nullity theorem: $\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(\text{domain of } T)$.

$\longmapsto$

Justifying the bases for Ker and Im

Ker:
$$\begin{pmatrix} -1 & 1 & 2 \\ -2 & 2 & 4 \\ -3 & 3 & 6 \end{pmatrix} \overset{\text{Gauss elim}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\left(A \,\middle|\, \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix}\right)$ has solutions $\left\{ \begin{pmatrix} -t-2s \\ t \\ s \end{pmatrix} : t,s \in \mathbb{R} \right\} \rightsquigarrow$ Take $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

Note that the non-pivot columns in rref(A) are 2 and 3. The entries 2 and 3 in the vectors we chose are 1 or 0: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. This ensures that they are linearly independent:

$$\lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Downarrow$$
$$\begin{pmatrix} -\lambda-2\mu \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda = \mu = 0.$$

This works in general: the vectors associated to the free variables always form a basis of Ker(T).

Im:

Theorem 1: Let $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ be a matrix and let $\mathrm{rref}(A) = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix}$.

Then, $\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \Leftrightarrow \lambda_1 w_1 + \dots + \lambda_n w_n$.

<u>Proof</u>: It suffices to show that if we have a linear dependence $\lambda_1 v_1 + \dots + \lambda_n v_n$, then apply a row operation and get $\begin{pmatrix} | & & | \\ v_1' & \cdots & v_n' \\ | & & | \end{pmatrix}$, then $\lambda_1 v_1' + \dots + \lambda_n v_n' = 0$.

- Swap two rows: $\lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$

$$\Downarrow \; \mathrm{I} \leftrightarrow \mathrm{II}$$

$$\lambda_1 \begin{pmatrix} a_{21} \\ a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{2n} \\ a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0 \quad \checkmark \quad \text{(other rows: same story)}$$

- Multiply a row by λ: $\lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$

$$\Downarrow$$

$$\lambda_1 \begin{pmatrix} \lambda a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda \begin{pmatrix} \lambda a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0 \quad \checkmark \quad \text{(other rows: same story)}$$

- Add a multiple of a row to another: $\lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$

$$\Downarrow \; \mathrm{I} \to \mathrm{I} + \lambda \mathrm{II}$$

$$\lambda_1 \begin{pmatrix} a_{11} + \lambda a_{21} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + \lambda \begin{pmatrix} a_{1n} + \lambda a_{2n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$$

This shows that "row operations respect linear dependences", so the linear dependences of A are those of $\mathrm{rref}(A)$. □

Application to Im: the columns in $\mathrm{rref}(A)$ with a pivot will be l.i. and span the other columns in $\mathrm{rref}(A)$

$\Rightarrow$ A basis for $\mathrm{Im}(A)$ is given by the columns in A corresponding to pivot columns in $\mathrm{rref}(A)$.

<u>Justifying that all bases of a subspace have the same number of vectors</u> (Warm up for next week).

<u>Theorem 2</u>: Let $S \subseteq \mathbb{R}^n$ be a subspace and let $v_1, \dots, v_m$ be a basis for S. Then any $s \in S$ can be written in a <u>unique way</u> as $s = \lambda_1 v_1 + \dots + \lambda_m v_m$.

<u>Proof</u>: The fact that $s = \lambda_1 v_1 + \dots + \lambda_m v_m$ for <u>some</u> $\lambda_1, \dots, \lambda_m$ is obvious since $v_1, \dots, v_m$ span S.

Uniqueness: Suppose $\lambda_1 v_1 + \dots + \lambda_m v_m = \mu_1 v_1 + \dots + \mu_m v_m$ are two ways to write s as a linear combination of $v_1, \dots, v_m$. Then $(\lambda_1 - \mu_1) v_1 + \dots + (\lambda_m - \mu_m) v_m = 0 \underset{v_i \text{ are l.i.}}{\Longrightarrow} \lambda_1 - \mu_1 = \dots = \lambda_m - \mu_m = 0$

$\Rightarrow \lambda_i = \mu_i$ for all i. □

Theorem 3: Any two bases of a subspace $S \subseteq \mathbb{R}^n$ have the same number of vectors.

<u>Proof</u>: Let $v_1, \dots, v_m$ be one basis and let $w_1, \dots, w_k$ be another basis.

Then w_i can be written uniquely as $\lambda_{i1} v_1 + \dots + \lambda_{im} v_m$ for each $i = 1, \dots, k$.

v_i can be written uniquely as $\mu_{i1} w_1 + \dots + \mu_{ik} w_k$ for each $i = 1, \dots, n$.

Consider the linear transformations $T_1 : \mathbb{R}^m \to \mathbb{R}^k$ with matrix $A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \vdots & & & \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{km} \end{pmatrix}$

$T_2 : \mathbb{R}^k \to \mathbb{R}^m$ with matrix $B = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1k} \\ \vdots & & & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mk} \end{pmatrix}$

Claim: A and B are inverse to each other, therefore square, therefore $m = k$.

Proof of the claim:

If we write $v_1 = \mu_{11} w_1 + \dots + \mu_{1k} w_k$

$$= \mu_{11}(\lambda_{11} v_1 + \dots + \lambda_{1m} v_m) + \mu_{12}(\lambda_{21} v_1 + \dots + \lambda_{2m} v_m) + \dots + \mu_{1k}(\lambda_{k1} v_1 + \dots + \lambda_{km} v_m)$$

$$= \underbrace{(\mu_{11}\lambda_{11} + \mu_{12}\lambda_{21} + \dots + \mu_{1k}\lambda_{k1})}_{(\text{1st row of } A)\cdot(\text{1st column of } B)} v_1 + (\mu_{11}\lambda_{12} + \dots + \mu_{1k}\lambda_{k2}) v_2 + \dots + \underbrace{(\mu_{11}\lambda_{1m} + \dots + \mu_{1k}\lambda_{km})}_{(\text{last row of } A)\cdot(\text{1st column of } B)} v_m$$

Doing this for each v_i, we get

$$AB = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}. \quad \text{Similarly } BA = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \text{ and the claim follows.} \ \square$$

Assorted questions and solutions

2. If A is a 5×6 matrix of rank 4, then the nullity of A is 1. (True or false?)

$$5\left\{ \underbrace{\left(\qquad\qquad \right)}_{6} \right. : \mathbb{R}^6 \longrightarrow \mathbb{R}^5 \qquad \underbrace{\text{rank}}_{4} + \underset{\substack{\| \\ 2}}{\text{nullity}} = 6 \quad \Rightarrow \text{False!}$$

5. There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$. (True or false?)

False: $\dim(\mathrm{Im}(T)) = 5$, domain $= \mathbb{R}^4 \Rightarrow 4 = 5 + \underbrace{\dim(\mathrm{Ker})}_{\geq 0}$ Impossible.

18. If the image of an $n \times n$ matrix A is all of $\mathbb{R}^n$, then ~~A~~ the linear transf. associated to A must be ~~invertible~~. injective (therefore invertible)

True: $n = \underbrace{\dim(\mathrm{Im}(T))}_{n} + \underset{\substack{\Downarrow \\ 0}}{\dim(\mathrm{Ker}(T))}$

24. Vectors $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}, \begin{bmatrix}9\\8\\7\\6\end{bmatrix}, \begin{bmatrix}5\\4\\3\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\-2\end{bmatrix}$ are linearly independent.

False: no 5 l.i. vectors in $\mathbb{R}^4$

30. If vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then vector $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

False: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$.

35. If $AB = 0$ for two 2×2 matrices A and B, then BA must be the zero matrix as well.

False: $\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$.

Prove that

44. If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\mathrm{rank}(A) \leq 5$ must hold.

$A^2 = 0$: $\mathbb{R}^{10} \longrightarrow \mathbb{R}^{10} \longrightarrow \mathbb{R}^{10}$

this kills $\mathrm{Im}(A)$ i.e. $\mathrm{Im}(A) \subseteq \mathrm{Ker}(A)$. So $\dim(\mathrm{Im}(A)) \leq \dim(\mathrm{Ker}(A)) = 10 - \dim(\mathrm{Im}(A))$

$\Rightarrow 2\dim(A) \leq 10 \Rightarrow \dim(A) \leq 5$.

28. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\4\end{bmatrix}, \quad \begin{bmatrix}2\\3\\4\\k\end{bmatrix}$$

Want them to span $\mathbb{R}^4 \Rightarrow$ Want rank $\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\2&3&4&k\end{pmatrix} = 4$

$$\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&k-4-9-16\end{pmatrix} \Rightarrow k = 29$$

31. Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Find a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that $\ker(T) = \{\vec{0}\}$ and $\operatorname{im}(T) = V$. Describe T by its matrix A.

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -\frac{1}{4}(-x_1+x_2-2x_3) \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$\Rightarrow \begin{pmatrix}1\\0\\0\\1/4\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1/4\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1/2\end{pmatrix}$ span the subspace.

Claim: they are l.i.

Proof: $\lambda_1\begin{pmatrix}1\\0\\0\\1/4\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\\-1/4\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\1\\1/2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\frac{1}{4}\lambda_1 - \frac{1}{4}\lambda_2 + \frac{1}{2}\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \lambda_1, \lambda_2, \lambda_3 = 0$ ✓

$\Rightarrow$ They form a basis of V.

$T: \mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\1/4&-1/4&1/2\end{pmatrix}$$

Ker $= 0$ because they are l.i.

Im $= V$ because they span V

33. A subspace V of $\mathbb{R}^n$ is called a *hyperplane* if V is defined by a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0,$$

where at least one of the coefficients c_i is nonzero. What is the dimension of a hyperplane in $\mathbb{R}^n$? Justify your answer carefully. What is a hyperplane in $\mathbb{R}^3$? What is it in $\mathbb{R}^2$?

Consider $T: \mathbb{R}^n \to \mathbb{R}$. Then $V = \mathrm{Ker}(T)$.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto c_1x_1 + \cdots + c_nx_n$$

Since some $c_i \neq 0$, $\mathrm{Im}(T) = \mathbb{R}$. Then $n = \dim(\underbrace{\mathrm{Ker}(T)}_{V}) + \underbrace{\dim(\mathbb{R})}_{1} \Rightarrow \dim(V) = n-1$.

Hyperplane in $\mathbb{R}^3$: a plane

in $\mathbb{R}^2$: a line.

In-class exercises:

30. Find a ~~basis of the subspace of $\mathbb{R}^4$ defined by the equation~~

Prove that $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{array}{l} 2x_1 - x_2 + 2x_3 + 4x_4 = 0. \\ x_1 + x_2 - x_3 - x_4 = 0 \end{array} \right\}$ is a subspace of $\mathbb{R}^4$, find a basis for it (You must show it is a basis).

37. Give an example of a 4×5 matrix A with $\dim(\ker A) = 3$.

36. Can you find a 3×3 matrix A such that $\mathrm{im}(A) = \ker(A)$? Explain.

Intuition: a linear transformation is a transformation

<u>Example 5</u> / Discussion

Recall a system of equations looks like:

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 2 \\
2x_1 - x_2 + x_3 &= 3 \\
-x_1 + x_2 + 3x_3 &= 4
\end{aligned}
$$

Writing both sides as vectors, we get

$$
\begin{pmatrix} x_1 + 2x_2 - x_3 \\ 2x_1 - x_2 + x_3 \\ -x_1 + x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}
$$