Lecture 7:

In-class exercises from last time:

1. Find the image and kernel of the transformations associated to the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

For A, the image is $\mathrm{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right)$. I claim that this is all of $\mathbb{R}^2$.

We check this by computing the rank of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ (see lecture 4).

Indeed $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{Gauss elim}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \mathrm{rank}(A) = 2 \Rightarrow \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$ are a spanning set of $\mathbb{R}^2$.

Therefore the image of the first transformation is $\mathbb{R}^2$.

Kernel: $\mathrm{Ker}(T) = \{v \in \mathbb{R}^3 : Av = 0\}$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix} x+y+z \\ x+2y+3z \end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\}$$

↳ solving this system gives $x = t, y = -2t, z = t$

$$= \left\{ \begin{pmatrix}t\\-2t\\t\end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left(\begin{pmatrix}1\\-2\\1\end{pmatrix}\right).$$

For B, the image is $\mathrm{Span}\left(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right)$

Notice that this is "redundant": $\begin{pmatrix}2\\-1\\1\end{pmatrix} = -2 \cdot \begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$

So the image is $\mathrm{Span}\left(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right)$

Kernel: $\mathrm{Ker}(T) = \{v \in \mathbb{R}^3 : Bv = 0\}$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \right\}$$

↳ system gives...

$$= \left\{ \begin{pmatrix}2t\\-t\\t\end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left(\begin{pmatrix}2\\-1\\1\end{pmatrix}\right).$$

not a coincidence! We explain this today.

2. If $s_1, s_2 \in Ker(T)$, then $T(s_1+s_2) \underset{\substack{\downarrow \\ T \text{ linear}}}{=} T(s_1) + T(s_2) = 0 \Rightarrow s_1+s_2 \in Ker(T)$.

If $s \in Ker(T)$, $\lambda \in \mathbb{R}$, then $T(\lambda s) \underset{\substack{\downarrow \\ T \text{ linear}}}{=} \lambda T(s) = 0 \Rightarrow \lambda s \in Ker(T)$.

3. $Im(T) = Span\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) \overset{(1)}{\Rightarrow}$ Columns are multiples of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$Ker(T) = Span\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \overset{(2)}{\Rightarrow}$ $1 \cdot$ first column $+ 2 \cdot$ second column $= 0$.

So $\underset{(1)}{\text{take}}$ $\begin{pmatrix} 3 & a \\ 4 & b \end{pmatrix} \underset{(2)}{\Rightarrow} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2a+3=0 & \Rightarrow a = -\frac{3}{2} \\ 2b+4=0 & \Rightarrow b = -\frac{1}{2} \end{matrix}$

So $\begin{pmatrix} 3 & -\frac{3}{2} \\ 4 & -\frac{1}{2} \end{pmatrix}$ may work: $Im(T)$ is definitely $Span\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right)$

$Ker(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} 3 & -\frac{3}{2} \\ 4 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\underset{\text{Gauss elim}}{=} Span\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

Recap:
- Defined injectivity, surjectivity
- Defined Ker, Im, subspaces in general.

Today: more on subspaces, dimension, rank-nullity theorem.

Discussion: we have discussed bases for $\mathbb{R}^n$, but we can generalize this idea to subspaces.

Definition 1: Let $S \subseteq \mathbb{R}^n$ be a linear subspace and let $v_1, \dots, v_m$ be vectors in S. Then $s_1, \dots, s_m$ form a basis of S iff:

- The vectors $v_1, \dots, v_m$ are linearly independent
- The span of the vectors $v_1, \dots, v_m$ is all of S.

Example 1: Consider the subspace S of $\mathbb{R}^3$, where $S = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$. Then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of S: they are linearly independent: $\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \lambda \\ 0 \\ \mu \end{pmatrix} = 0 \Rightarrow \lambda = \mu = 0$.

they span S: $S = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$

$= \left\{ x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : x, z \in \mathbb{R} \right\}$

$= Span\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.

The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$ also span S, but they are not l.i.

The vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is linearly independent (by itself), but it does not span S.

Remark: the theorem about maximality/minimality also holds for subspaces, and the proof is the same as the one we did for $\mathbb{R}^n$. We record it for completeness.

Theorem 1: A set of vectors $v_1, \dots, v_m$ is a "minimal" spanning set for S if and only if $v_1, \dots, v_m$ is a "maximal" linearly independent set in S. if and only if $v_1, \dots, v_m$ is a basis for S.

Question: do all bases have the same number of vectors?

Theorem 3: let $S \subseteq \mathbb{R}^n$ be a linear subspace. let $v_1, \dots, v_m$ be a basis for S, and let $w_1, \dots, w_k$ be another basis. Then $m = k$. In other words: any two bases of S have the same number of vectors.

Proof: Consider $\{v_1, v_2, \dots, v_{m-1}\}$. These do not span v_m (otherwise $v_1, \dots, v_m$ would be linearly dependent). So there is some w_i not spanned by $\{v_1, \dots, v_{m-1}\}$ (otherwise $v_1, \dots, v_{m-1}$ would span $w_1, \dots, w_k$, which in turn span S, but we showed they do not span v_m). Assume the w_i is actually w_n (we may reorder the w_i's so that this is true).

Next, consider $\{v_1, \dots, v_{m-1}, w_k\}$. Write $w_{k-1} = \lambda_1 v_1 + \dots + \lambda_{m-1} v_{m-1} + \mu_k w_k$

Then, one of the λ_i's is nonzero (otherwise w_{k-1} would be in $\mathrm{Span}(w_k)$, impossible). Assume it is λ_{m-1}.

Then, we can replace v_{m-1} by w_{k-1}. Repeat this process until replacing the v_i's by w_i's. Since one vector is replaced each time, it follows that $k \leq m$. Repeating this argument replacing w_i's by v_i's shows $m \leq k$. $\square$

We can now define:

Definition 2: The dimension of a subspace $S \subseteq \mathbb{R}^n$ is the number of vectors of (any) basis of S. We write it $\dim(S)$.

Example 2: Let $S \subseteq \mathbb{R}^3$ be $\left\{\begin{pmatrix}x\\0\\z\end{pmatrix} : x, z \in \mathbb{R}\right\}$. This has a basis $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$, hence $\dim(S) = 2$.

S also has a basis $\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$. To see this, note that $\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ are linearly independent, so their span is 2-dimensional, and lies inside $S \underset{\dim(S)=2}{\Rightarrow}$ it is a maximally linearly indep set in S

$\underset{\text{Thm 2}}{\Rightarrow}$ it is a basis

Remark: the dimension formalizes the idea that subspaces are $\underset{\dim 1}{\text{"a line"}}$, $\underset{\dim 2}{\text{"a plane"}}$, etc.

How to find bases for Im and Ker

Example 3:
$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}. \quad \longleftrightarrow \quad T: \mathbb{R}^5 \to \mathbb{R}^4$$

$$B = \operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Ker Recall that $\operatorname{Ker}(T) = \left\{ \text{solutions to } \left(A \,\middle|\, \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \right) \right\}$

$= \left\{ \text{solutions to } \left(\operatorname{rref}(A) \,\middle|\, \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \right) \right\}$

Now the solutions are $\left\{ \begin{pmatrix} -2t-3s+4r \\ t \\ 4s-5r \\ s \\ r \end{pmatrix} : t,s,r \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right).$

each free variable corresponds to a basis vector

Upshot: $\dim(\operatorname{Ker}(T)) = \#$ non-pivot columns in $\operatorname{rref}(A) = 3$. (= "nullity")

Im Recall that $\operatorname{Im}(T)$ is the span of the columns of A, but that some columns may be linear combinations of others. Key observation: columns a_i form a linear dependence if and only if the corresponding columns of B do.

It is now easy to see that
$$B = \operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

are linear combinations of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

It follows that a basis for $\operatorname{Im}(T)$ is the first and third vectors in A:

$\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 5 \\ 1 \end{pmatrix}$ form a basis for $\operatorname{Im}(T)$

Upshot: $\dim(\operatorname{Im}(T)) = \#$ pivot columns in $\operatorname{rref}(A) = \operatorname{rank}(A)$

Since #pivot columns + # non-pivot columns = #columns of A = dim(domain), we have arrived at the following theorem:

<u>Theorem 4</u> (Rank-nullity Theorem): Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$\dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T)) = n.$$

Example 4: $\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has kernel $\operatorname{Span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right) \to$ dim 1

image $\operatorname{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right) \to$ dim 2

$\Big\}$ add up to $3 = \dim(\mathbb{R}^3)$.

In-class exercises:

1. Compute the image of the transformation with matrix $\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, by finding a basis for it.

2. Compute the kernels and images of the transformations in Lecture 5, and verify the rank-nullity theorem in each case.

3. Find a basis for the subspace $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x+y+z=0 \right\} \subseteq \mathbb{R}^3$.