lecture 4	
Pecap: • Vectors, addition, scalar multiplication	
· Span, spanning set [ hasis	· · · · · · · · · · · · · · ·
· Linear independence	· · · · · · · · · · · · · · · ·
Today: More on span and linear dependence + Intro to finear	r transformations (Pre-class quiz)
Discussion: In the exercise session, we have seen that cl	hecking linear independence /spanning
amounts to solving systems of linear equations. Let	us make this precise.
Remark: I will often write O for (?) when there's no	ambiguity.
Linear independence revisited (an) (an)	
Suppose we have vectors $V_{4} = \begin{pmatrix} a_{a_1} \\ a_{m_{a_{m_{a}}}} \end{pmatrix}_{1,,N_{n}} = \begin{pmatrix} a_{a_{n}} \\ a_{m_{n}} \end{pmatrix}_{1n} \mathbb{R}^{m}$ , and we	e want to know if they are linearly independent.
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{21} \\ a_{m2} \end{pmatrix}, \dots, v_{n} = \begin{pmatrix} a_{2n} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can   find $x_{4}, x_{2}, \dots, x_{m}$ such that $x_{2}v_{4} + x_{4}v_{2} + \dots + v_{m}v_{m}$	+ xm vm = 0? If the only possibility is
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{a1} \\ a_{m2} \end{pmatrix}_{1,,} V_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}_{1n} \mathbb{R}^{m}$ , and we The question is: can I find $x_{4}, x_{2},, x_{m}$ such that $x_{1}v_{4} + x_{4}v_{5} + + \frac{1}{2}$ if $x_{1} = x_{2} = = x_{m} = 0$ , then $v_{4},, v_{m}$ will be linearly independent	t. Let's expand this out. We're looking
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{21} \\ a_{mn} \end{pmatrix}_{j,,j} V_{n} = \begin{pmatrix} a_{2n} \\ a_{mn} \end{pmatrix}_{jn} R^{m}$ , and we The question is: can I find $x_{4}, x_{2},, x_{m}$ such that $x_{2}v_{4} + x_{4}v_{5} + + I$ if $x_{4} = x_{2} = = x_{m} = 0$ , then $v_{4},,v_{m}$ will be linearly independent for $x_{4},,x_{m}$ such that:	t want to know if they are linearly independent. $+ \times m \vee m = 0$ ? If the only possibility is t. Let's expand this out. We're looking
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{21} \\ a_{m2} \end{pmatrix}_{1,,3} v_{n} = \begin{pmatrix} a_{2n} \\ a_{mn} \end{pmatrix}_{1n} R^{m}$ , and we The question is: can   find $x_{4}, x_{2},,x_{m}$ such that $x_{1}v_{4} + x_{2}v_{2} + + I$ if $x_{1} = x_{2} = = x_{m} = 0$ , then $v_{4},,v_{m}$ will be linearly independent for $x_{4},,x_{m}$ such that: $\begin{pmatrix} a_{42} \\ a_{2} \end{pmatrix}_{1,,3} \begin{pmatrix} a_{42} \\ a_{22} \end{pmatrix}_{1,,3} \begin{pmatrix} a_{4n} \\ a_{2n} \end{pmatrix}$	t want to know if they are linearly independent. $t \times m \vee m = 0$ ? If the only possibility is t. Let's expand this out. We're booking
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{a1} \\ a_{m2} \end{pmatrix}_{1,,3} v_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}_{1n} R^{m}$ , and we The question is: can   find $x_{4}, x_{2},, x_{m}$ such that $x_{1}v_{4} + x_{4}v_{4} + + t_{n}$ if $x_{4} = x_{2} = = x_{m} = 0$ , then $v_{4},, v_{m}$ will be linearly independent for $x_{4},, x_{m}$ such that: $0 = x_{4}v_{4} + + x_{m}v_{m} = x_{4}\begin{pmatrix} a_{4} \\ a_{3} \\ a_{m1} \end{pmatrix}_{1} + x_{4}\begin{pmatrix} a_{4} \\ a_{2} \\ a_{m2} \end{pmatrix}_{1} + + x_{m}\begin{pmatrix} a_{4} \\ a_{2} \\ a_{mn} \end{pmatrix}_{1}$	t want to know if they are linearly independent. $+x_m v_m = 0$ ? If the only possibility is t. Let's expand this out. We're booking
Suppose we have vectors $v_{4} = \begin{pmatrix} a_{a1} \\ a_{m2} \end{pmatrix}$ , $v_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can   find $x_{4}, y_{2}, \dots, y_{m}$ such that $x_{2}y_{2} + x_{4}y_{4} + \dots + y_{4}y_{4} + \dots + y_{4}y$	t want to know if they are tinearity independent. $t \times m \vee m = 0$ ? If the only possibility is t. Let's expand this out. We're booking
Suppose we have vectors $v_{3} = \begin{pmatrix} a_{21} \\ a_{11} \\ a_{m2} \end{pmatrix}$ , $v_{n} = \begin{pmatrix} a_{2n} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can $\int \int a_{mn} x_{2} x_{2} x_{2} x_{m}$ such that $x_{2}v_{2} + x_{2}v_{2} + \dots + v_{n}v_{n} = 0$ , then $v_{2}, \dots, v_{m}$ will be linearly independent for $x_{2}, \dots, x_{m}$ such that: $\int x_{1} = x_{2} = \dots = x_{m} = 0$ , then $v_{2}, \dots, v_{m}$ will be linearly independent for $x_{2}, \dots, x_{m}$ such that: $\int a_{m1} x_{1} + x_{2} \begin{pmatrix} a_{2n} \\ a_{2n} \\ a_{mn} \end{pmatrix} + x_{n} \begin{pmatrix} a_{2n} \\ a_{2n} \\ a_{mn} \end{pmatrix} + \dots + x_{m} \begin{pmatrix} a_{2n} \\ a_{2n} \\ a_{mn} \end{pmatrix}$ $= \begin{pmatrix} a_{2n}x_{1} + a_{2n}x_{2} + \dots + a_{2n}x_{n} \\ a_{2n}x_{2} + \dots + a_{2n}x_{n} \\ a_{2n}x_{2} + \dots + a_{2n}x_{n} \end{pmatrix}$ (k)	+ Xm Vm = 0? If the only possibility is t. Let's expand this out. We're booking
Suppose we have vectors $v_{\underline{a}} = \begin{pmatrix} a_{\underline{a}1} \\ a_{\underline{a}nz} \end{pmatrix}$ , $v_n = \begin{pmatrix} a_{\underline{a}n} \\ a_{\underline{a}nn} \end{pmatrix}$ in $\mathbb{R}^m$ , and we The question is: can $  \int     d      d     d     d     d     d  $	+ xm vm = 0? If the only possibility is t. Let's expand this out. We're booking matrix
Suppose we have vectors $v_{a} = \begin{pmatrix} a_{a1} \\ a_{ma} \end{pmatrix}$ , $v_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can $  \int     d      d     d      d      d      d      d      d       $	+ xm vm = 0? If the only possibility is t. Let's expand this out. We're booking matrix
Suppose we have vectors $v_{a} = \begin{pmatrix} a_{a1} \\ a_{max} \end{pmatrix}$ , $v_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can $  \int    d    x_{a}, v_{a},, v_{m} $ such that $x_{a}v_{a} + x_{a}v_{a} + + 1$ $\int x_{a} = x_{a} = = x_{m} = 0$ , then $v_{a},, v_{m} $ will be linearly independent for $x_{a},, x_{m} $ such that: $O = x_{a}v_{a} + + x_{m}v_{m} = x_{a}\begin{pmatrix} a_{aa} \\ a_{a1} \\ a_{ma} \end{pmatrix} + x_{a}\begin{pmatrix} a_{a2} \\ a_{22} \\ a_{ma} \end{pmatrix} + + x_{m}\begin{pmatrix} a_{am} \\ a_{m} \\ a_{mn} \end{pmatrix}$ $= \begin{pmatrix} a_{aa}x_{a} + a_{a2}x_{a} + + a_{am}x_{m} \\ a_{ma}x_{a} + a_{a2}x_{a} + + a_{am}x_{m} \\ a_{ma}x_{a} + a_{ma}x_{a} + + a_{mm}x_{m} \end{pmatrix}$ (k) In other words, we need to solve the system with asymptoted $\begin{pmatrix} a_{a1} & a_{a2} & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \end{pmatrix}$	+xmvm = 0? If the only possibility is t. Let's expand this out. We're booking waterix
Suppose we have vectors $v_{a} = \begin{pmatrix} a_{a1} \\ a_{m2} \end{pmatrix}$ ,, $v_{n} = \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^{m}$ , and we The question is: can $ \int ind x_{a}, y_{a},, x_{m}$ such that $x_{va} + x_{a}v_{a} + + i$ $\int x_{a} = x_{a} = = x_{m} = 0$ , then $v_{a},, v_{m}$ will be linearly independent for $x_{a},, x_{m}$ such that: $0 = x_{a}v_{a} + + x_{m}v_{m} = x_{a} \begin{pmatrix} a_{aa} \\ a_{a1} \\ a_{m1} \end{pmatrix} + x_{a} \begin{pmatrix} a_{a2} \\ a_{22} \\ a_{m2} \end{pmatrix} + + x_{m} \begin{pmatrix} a_{an} \\ a_{mn} \end{pmatrix}$ $= \begin{pmatrix} a_{aa}x_{a} + a_{a2}x_{a} + + a_{an}x_{n} \\ a_{a}x_{a} + a_{a2}x_{a} + + a_{am}x_{n} \\ a_{m2}x_{a} + a_{ma}x_{a} + + a_{mm}x_{n} \end{pmatrix}$ (k) In other words, we need to solve the system with argumented $\begin{pmatrix} a_{11} & a_{12} & & a_{mn} \\ a_{m2} & & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	+ xm Vm = 0? If the only possibility is t. Let's expand this out. We're booking watrix

Observe that the system is consistent: $x_a =$ The question is: are there infinitely many?	$x_{e} = \dots = x_{m} = 0$ is a solution
are there free variables in v =>	
ls there a pivot per colle	omn?
Is rank $(A) = n = \#$ vector in short, we have the following: The following:	
(represent 1: the vectors $V_3 = \begin{pmatrix} u_{g1} \\ a_{ma} \end{pmatrix}$ ,, $V_n = \begin{pmatrix} u_{gn} \\ a_{mn} \end{pmatrix}$	$ (a_{a_{1}}, a_{n}, a_{n}, a_{n}) = n $
A similar story holds for the spanning The question is for any $w = \begin{pmatrix} b_2 \\ b_m \end{pmatrix}$ , can I find $x_2, x_3$	property. $x_1, x_2, x_3, x_4, x_5, x_5, x_5, x_5, x_5, x_5, x_5, x_5$
Let's write it out again. This time we get: $\begin{pmatrix} a_{34}x_1 + a_{32}x_2 + \dots + a_{3n}x_n \\ a_{31}x_2 + a_{22}x_2 + \dots + a_{2n}x_n \\ a_{ma}x_1 + a_{ma}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$	· · · · · · · · · · · · · · · · · · ·
The question becomes is the system	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
Does rief (M) have no rows like	(O

$\longleftrightarrow \ \vdots \ $
Does rief(A) have a pivot in each row?
ls rank (A) = $m \in \#vanables$ )?
We now have:
Theorem 2: the vectors $v_a = \begin{pmatrix} a_{a,a} \\ a_{a,i} \\ \dots \end{pmatrix} = \begin{pmatrix} a_{a,n} \\ a_{a,n} \end{pmatrix}$ in $\mathbb{R}^m$ are a spanning set $\iff$ rank $(A) = m$ .
$\begin{pmatrix} a_{12} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n3} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & \vdots & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & \vdots & a_{n4} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots \\ a_{n4} & \vdots & \vdots \\ a_{n4} & \vdots & a_{n4} & \cdots & a_{n4} \\ \vdots & \vdots & \vdots \\ a_{n4} & \vdots & \vdots \\ a_{n4} & \vdots & a_{n4} & \vdots \\ a_{n4} & \vdots & \vdots \\ a_{n4} & \vdots & a_{n4} & \vdots$
Linear transformations
Inspired by the linear combinations story, we define a linear transformation as follows:
<u>Definition 1</u> : A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T:\mathbb{R}^n \to \mathbb{R}^m$
of the form $ \begin{pmatrix} X_{1} \\ X_{2} \\ \vdots \\ X_{m} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} X_{1} + a_{12} X_{2} + \dots + a_{1n} X_{n} \\ a_{21} X_{2} + a_{22} X_{2} + \dots + a_{2n} X_{n} \\ \vdots \\ a_{m_{2}} X_{2} + a_{m_{2}} X_{2} + \dots + a_{mn} X_{n} \end{pmatrix} $ (*)
(for some fixed values a;;)
The matrix of the transformation is $\begin{pmatrix} a_{11} & \cdots & a_{nn} \\ \vdots & \vdots \\ a_{m_1} & \cdots & a_{m_m} \end{pmatrix}$
forming the vector (;), we cerine mainix-vector multiplication as
$A_{x} = \begin{pmatrix} a_{11}x_{1} + a_{12}x_{2} + \dots + a_{2n}x_{n} \\ a_{21}x_{2} + a_{22}x_{2} + \dots + a_{2n}x_{n} \\ \vdots \\ a_{m2}x_{2} + a_{m2}x_{2} + \dots + a_{mn}x_{n} \end{pmatrix}$
Pennauk: Matrix-vector multiplication is often introduced without motivation. One important way
to think about it is the linear combination interpretation: a matrix $A = (v_1 \dots v_n)$
sends $\begin{pmatrix} \lambda_1 \\ \lambda_n \end{pmatrix}$ to the linear combination $\lambda_1 v_1 + \dots + \lambda_n v_n$ .

$\underbrace{\text{Example 1}}_{2-3} \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 - 1 \cdot 1 \\ 2 \cdot 4 - 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ -3 \end{pmatrix}$
$\underbrace{\text{Example 2}}_{2 -1 2} \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$
Example 3: (Important observation) $\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{-3} \end{pmatrix}$
In words the image of $c$ is the first column, the image of $j$ is the second column.
Important property of linear transformations:
Theorem 3: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $v, w \in \mathbb{R}^n$ . Then,
$A = T(y_{+}y_{+}) - T(y_{+}) + T(y_{+})$
$(\psi + \psi) = ((\psi + \psi)) = ((\psi) + ((\psi)) = ((\psi) + (\psi)) = ((\psi) + (\psi) + $
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ Proof 1) let $A = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}$ be the matrix associated to the linear transformation.
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ $Proof: 1)$ let $A = \begin{pmatrix} a_{1} & \dots & a_{n} \\ 0 & \dots & 0 \end{pmatrix}$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{1} \\ v_{n} \end{pmatrix}$ , $w = \begin{pmatrix} w_{1} \\ w_{n} \end{pmatrix}$
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ Proof 1) let $A = \begin{pmatrix} 1 & & & \\ a_{2} & - & a_{n} \\ 1 & & 1 \end{pmatrix}$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{2} \\ v_{n} \end{pmatrix}$ , $w = \begin{pmatrix} w_{2} \\ w_{n} \end{pmatrix}$ . Then $T(v+w) = T\begin{pmatrix} v_{2}+w_{2} \\ w_{n} \end{pmatrix}$
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ Proof 1) let $A = \begin{pmatrix} a_{1} & \dots & a_{n} \\ a_{n} & \dots & a_{n} \end{pmatrix}$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{1} \\ v_{n} \end{pmatrix}$ , $w = \begin{pmatrix} w_{1} \\ w_{n} \end{pmatrix}$ . Then $T(v+w) = T\begin{pmatrix} v_{1}+w_{2} \\ w_{n} \end{pmatrix}$ $= (v_{2}+w_{2}) a_{1} + (v_{2}+w_{2}) a_{2} + \dots + (v_{n}+w_{n}) a_{n}$
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ Proof: 1) let $A = \begin{pmatrix} 1 & 1 & 1 \\ a_{2} & - & a_{n} \\ 1 & 1 \end{pmatrix}$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{1} \\ v_{n} \end{pmatrix}$ , $w = \begin{pmatrix} w_{1} \\ w_{n} \end{pmatrix}$ Then $T(v+w) = T\begin{pmatrix} v_{1}+w_{n} \\ \vdots \\ v_{n}+w_{n} \end{pmatrix}$ $= (v_{1}+w_{1}) a_{1} + (v_{2}+w_{2}) a_{2} + \cdots + (v_{n}+w_{n}) a_{n}$ $= (v_{2}a_{1} + \cdots + v_{n}a_{n}) + (w_{1}a_{2} + \cdots + w_{n}a_{n})$
2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ $Proof(\Delta v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ $Proof(\Delta v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$ $Proof(\Delta v) = \left(\begin{array}{c} a_{\pm} & - & a_{\pm} \\ a_{\pm} & - & a_{\pm} \\ a_{\pm} & - & a_{\pm} \end{array}\right)$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{\pm} \\ v_{\pm} \\ v_{\pm} \end{pmatrix}$ , $w = \begin{pmatrix} w_{\pm} \\ w_{\pm} \\ w_{\pm} \end{pmatrix}$ . $Then T(v+w) = T \begin{pmatrix} v_{\pm} + w_{\pm} \\ v_{\pm} + w_{\pm} \end{pmatrix}$ $= (v_{\pm} + u_{\pm}) a_{\pm} + (v_{\pm} + w_{\pm}) a_{\pm} + \cdots + (v_{\pm} + w_{\pm}) a_{\pm}$ $= (v_{\pm} a_{\pm} + \cdots + v_{\pm} a_{\pm}) + (u_{\pm} a_{\pm} + \cdots + w_{\pm} a_{\pm})$ = T(v) + T(w).
2) $T(\lambda v) = \lambda T(v)$ for all salars $\lambda \in \mathbb{R}$ Proof 1)Let $A = \begin{pmatrix} a_{1} & \dots & a_{n} \\ 1 & \dots & 1 \end{pmatrix}$ be the matrix associated to the linear transformation. and write $v = \begin{pmatrix} v_{n} \\ v_{n} \end{pmatrix}$ , $w = \begin{pmatrix} w_{1} \\ w_{n} \end{pmatrix}$ . Then $T(v+w) = T\begin{pmatrix} v_{1}+w_{n} \\ v_{n}+w_{n} \end{pmatrix}$ $= (v_{1}+v_{1}) a_{1} + (v_{2}+w_{2}) a_{2} + \dots + (v_{n}+w_{n}) a_{n}$ $= (v_{2}a_{2} + \dots + v_{n}a_{n}) + (w_{k}a_{k} + \dots + w_{n}a_{n})$ = T(v) + T(w). 2) Exercise.

One may wonder if the conditions in Theorem 3 imply that T is linear.  
The following theorem says that the answer is yes.  
Theorem 4: let T: R<sup>n</sup> 
$$\rightarrow$$
 R<sup>m</sup> be a function satisfying:  $T(v_1, w) = T(v_1) + T(w)$   
and  $T(\lambda v) = \lambda T(v)$  for all  $v_1, w \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . Then T is a finar transformation,  
with associated matrix:  $\left(\frac{1}{1+1}, \dots, \frac{1}{1+1}, \dots\right)$ .  
Theory: Take the anomical basis  $e_1 = \left(\frac{4}{5}\right), \dots, e_n = \left(\frac{9}{2}\right)$ , and let  
 $a_1 = \left(\frac{6a_1}{a_1}\right), \dots, a_n = \left(\frac{6a_1}{a_m}\right)$  be  $T(e_1), \dots, T(e_n)$ , respectively.  
Then, for a given  $x = \left(\frac{6a_1}{x_n}\right)$ ,  
 $T\left(\frac{x_1}{a_1}\right) = T(x_1e_1 + \dots + x_n) = x_n T(e_1) + \dots + x_n T(e_n) = \left(\frac{a_1v_1 + a_1v_2v_1 + \dots + a_nv_n}{a_nv_n + \dots + a_nv_n}\right)$   
This shows that T is a linear transformation, with associated matrix A, as desired.  
In the following multiplications:  $\left(\frac{4}{0} - \frac{1}{2}\right) \begin{pmatrix} 2\\-1\\1 \end{pmatrix}$ ,  $\left(\frac{4}{1} - \frac{2}{3}\right) \begin{pmatrix} 4\\2\\2\\-1 \end{pmatrix}$ .  
2. Express the statement. "the vector  $\left(\frac{4}{3}\right)$  is a linear combination of the vectors  
 $\left(\frac{4}{1}, \left(\frac{9}{4}\right), \left(\frac{9}{2}\right)\right)$  with scalars  $4, 2, 2$ " with a modematical statement of the form  $Av = b$ .  
3. Prove theat a linear transformation  $T \cdot \mathbb{R}^n \to \mathbb{R}^n$  satisfies  $T(Av) = \lambda T(v)$   
for all v \in \mathbb{R}^n and  $\lambda \in \mathbb{R}$ .