Lecture 4

Recap: $\left.\begin{array}{l}\bullet\ \text{Vectors, addition, scalar multiplication}\\ \bullet\ \text{Span, spanning set}\\ \bullet\ \text{Linear independence.}\end{array}\right\}\text{basis}$

Today: More on span and linear dependence + Intro to linear transformations (Pre-class quiz)

Discussion: In the exercise session, we have seen that checking linear independence / spanning amounts to solving systems of linear equations. Let us make this precise.

Remark: I will often write 0 for $\begin{pmatrix}0\\\vdots\\0\end{pmatrix}$ when there's no ambiguity.

Linear independence revisited

Suppose we have vectors $v_1 = \begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{pmatrix}, \dots, v_n = \begin{pmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{pmatrix}$ in $\mathbb{R}^m$, and we want to know if they are linearly independent.

The question is: can I find $x_1, x_2, \dots, x_m$ such that $x_1v_1 + x_2v_2 + \dots + x_mv_m = 0$? If the only possibility is if $x_1 = x_2 = \dots = x_m = 0$, then $v_1, \dots, v_m$ will be linearly independent. Let's expand this out. We're looking for $x_1, \dots, x_m$ such that:

$$0 = x_1v_1 + \dots + x_mv_m = x_1\begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{pmatrix} + x_2\begin{pmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{pmatrix} + \dots + x_n\begin{pmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{pmatrix}$$

$$= \begin{pmatrix}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n\\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n\end{pmatrix} \qquad (*)$$

In other words, we need to solve the system with augmented matrix

$$\left(\begin{array}{c|c}\underbrace{\begin{matrix}a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \ddots & & \vdots\\ a_{m1} & \cdots & & a_{mn}\end{matrix}}_{A} & \begin{matrix}0\\0\\\vdots\\0\end{matrix}\end{array}\right)$$

Observe that the system is consistent: $x_1 = x_2 = \ldots = x_m = 0$ is a solution.

The question is: are there infinitely many?

$\Leftrightarrow$

are there free variables in $\mathrm{rref}(A)$?

$\Leftrightarrow$

Is there a pivot per column?

$\Leftrightarrow$

Is $\mathrm{rank}(A) = n \left(\begin{matrix} = \# \text{ vectors} \\ = \# \text{ columns} \end{matrix} \right)$?

In short, we have the following:

Theorem 1: the vectors $v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^m$ are linearly independent $\Leftrightarrow$ $\mathrm{rank}(\underset{\overset{\shortparallel}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}}}{A}) = n$.

Spanning set revisited

A similar story holds for the spanning property.

The question is: for any $w = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, can I find $x_1, x_2, \ldots, x_m$ such that $x_1 v_1 + x_2 v_2 + \ldots + x_m v_m = w$?

Let's write it out again. This time we get:

$$\begin{pmatrix} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$

The question becomes: is the system $\underbrace{\left(\begin{array}{c|c} \overbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{matrix}}^{A} & \begin{matrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{matrix} \end{array} \right)}_{M}$ compatible for all choices of b_i ?

$\Leftrightarrow$

Does $\mathrm{rref}(M)$ have no rows like $(0 \ldots 0 \mid b_i')$?

(subtlety: can we make b_i' nonzero? the answer is yes, don't focus on this for now.)

$\Longleftrightarrow$

Does rref(A) have a pivot in each row?

$\Longleftrightarrow$

Is $\mathrm{rank}(A) = m$ (= #variables) ?

We now have:

Theorem 2: the vectors $v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, v_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ in $\mathbb{R}^m$ are a spanning set $\Leftrightarrow$ $\mathrm{rank}(A) = m$.

$$\overset{\|}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}}$$

Linear transformations

Inspired by the linear combinations story, we define a linear transformation as follows:

Definition 1: A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \longmapsto \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} \qquad (*)$$

(for some fixed values a_{ij})

The matrix of the transformation is $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Forming the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, we define matrix-vector multiplication as

$$Ax = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

Remark: Matrix-vector multiplication is often introduced without motivation. One important way to think about it is the linear combination interpretation: a matrix $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ sends $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ to the linear combination $\lambda_1 v_1 + \dots + \lambda_n v_n$.

Example 1: $\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 4 - 1\cdot 1 \\ 2\cdot 4 - 3\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

Example 2: $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

Example 3: (Important observation)

$$\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 \\ 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 \\ 2\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

In words the image of $\hat{\imath}$ is the first column, the image of $\hat{\jmath}$ is the second column.

Important property of linear transformations:

Theorem 3: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let $v, w \in \mathbb{R}^n$. Then,

1) $T(v+w) = T(v) + T(w)$

2) $T(\lambda v) = \lambda T(v)$ for all scalars $\lambda \in \mathbb{R}$.

Proof: 1) Let $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$ be the matrix associated to the linear transformation.

and write $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

Then $T(v+w) = T \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$

$$= (v_1+w_1)\, a_1 + (v_2+w_2)\, a_2 + \cdots + (v_n + w_n)\, a_n$$

$$= (v_1 a_1 + \ldots + v_n a_n) + (w_1 a_1 + \ldots + w_n a_n)$$

$$= T(v) + T(w).$$

2) Exercise.

One may wonder if the conditions in Theorem 3 imply that T is linear. The following theorem says that the answer is yes.

<u>Theorem 4</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a function satisfying: $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$ for all $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then T is a linear transformation, with associated matrix $\begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_m) \\ | & & | \end{pmatrix}$.

<u>Proof</u>: Take the canonical basis $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, and let

$a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, a_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ be $T(e_1), T(e_2), \dots, T(e_n)$ respectively.

Then, for a given $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = T(x_1 e_1 + \dots + x_n e_n) \underset{\substack{\downarrow \\ \text{Assumption}}}{=} x_1 T(e_1) + \dots + x_n T(e_n) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}.$$

This shows that T is a linear transformation, with associated matrix A, as desired.

<u>In-class exercises</u>:

1. Perform the following multiplications: $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

2. Express the statement "the vector $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with scalars $1, 2, 2$" with a mathematical statement of the form $Av = b$.

3. Prove that a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfies $T(\lambda v) = \lambda T(v)$ for all $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.