Lecture 3

Change gears: vectors (Pre-class quiz)

Depending on who you ask, a vector is:

- An arrow in space (the physics student)
- A list of numbers (the CS student)
- An element in a vector space (the math student)

We will take the CS way, but be aware that these are all equivalent. Thus:

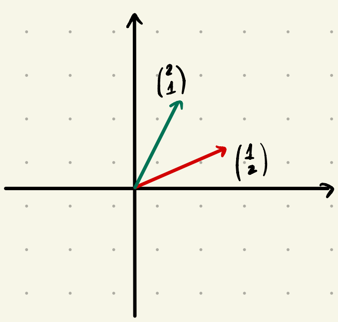
Definition 1: a vector v in $\mathbb{R}^n$ is a tuple of real numbers: $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$.

Definition 2: The sum of two vectors $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ is what you think it is: $v+w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$

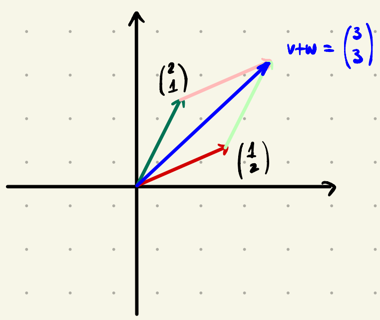
Definition 3: The multiplication of a vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ by a real number $\lambda \in \mathbb{R}$ (a "scalar") is also what you think it is: $\lambda v = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}$

Example 1: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a vector in $\mathbb{R}^2$, and so is $w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

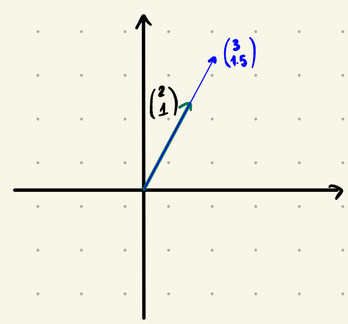
We can draw these as arrows from the origin to the respective coordinates.



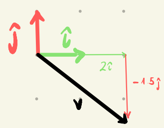
Their sum is then



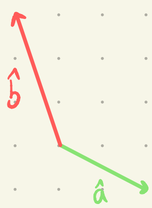
Taking $\lambda = 1.5$, $\lambda w = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$:

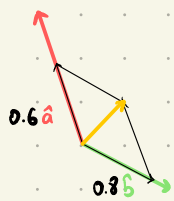


Discussion: consider the vectors $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Notice that we can scale $\hat{\imath}$ and $\hat{\jmath}$ and add them together to get <u>any</u> other vector. For instance, the vector $v = \begin{pmatrix} 2 \\ -1.5 \end{pmatrix}$ can be written as $2\hat{\imath} - 1.5\hat{\jmath}$.



Most of the time (in a sense that we're about to make precise), if you pick any two vectors, they will have this property: take $\hat{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\hat{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Then, for instance, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.8\hat{a} + 0.6\hat{b}$





In fact, one can write any vector as $\lambda\hat{a} + \mu\hat{b}$ for some $\lambda, \mu \in \mathbb{R}$. To find λ and μ, we just have to solve a system of linear equations!

$$\lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 3 & 1 \end{array}\right) \xrightarrow{I \to \frac{1}{2}I} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{II \to II + I} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow{II \to \frac{2}{5}II} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{5} \end{array}\right)$$

$$\xrightarrow{I \to I + \frac{1}{2}II} \left(\begin{array}{cc|c} 1 & 0 & \frac{4}{5} \\ 0 & 1 & \frac{3}{5} \end{array}\right)$$

<u>Definition 4</u>: A vector b is called a linear combination of the vectors $v_1, \dots, v_m$ iff it can be written as $w = \lambda_1 v_1 + \dots + \lambda_n v_n$ for some scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

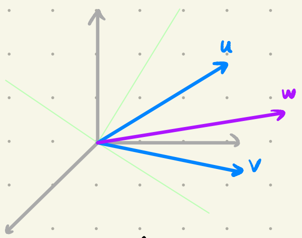
Example 2: The vector $\begin{pmatrix}1\\3\\5\end{pmatrix}$ is equal to $1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}+2\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$

so it is a linear combination of $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

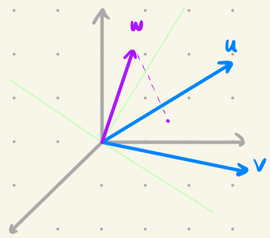
Example 3: The vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is <u>not</u> a linear combination of the vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Indeed, every linear combination will have to be of the form $\begin{pmatrix}*\\*\\0\end{pmatrix}$.

Example 4: Geometrically, a vector $w \in \mathbb{R}^3$ (in $\mathbb{R}^3$) is a linear combination of $u, v \in \mathbb{R}^3$ if and only if w lies in the plane containing "the arrows" u and v:



w is a linear combination of u, v



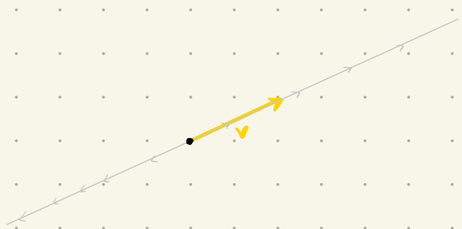
w is <u>not</u> a linear combination of u and v

Definition 5: Let $v_1, \dots, v_m$ be vectors in $\mathbb{R}^n$. Then the span of $v_1, \dots, v_m$ is the set

$$\{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m : \lambda_1, \dots, \lambda_m \in \mathbb{R}^n\}.$$

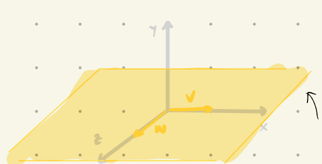
Example 4: The span of the vector $v = \begin{pmatrix}2\\1\end{pmatrix}$ in $\mathbb{R}^2$ is $\{\lambda v : \lambda \in \mathbb{R}\} = \{\begin{pmatrix}2\lambda\\\lambda\end{pmatrix} : \lambda \in \mathbb{R}\}$

Geometrically, it consists of all the vectors whose tip lies in the line containing v:



Example 5: The span of the vectors $v = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $w = \begin{pmatrix}0\\0\\1\end{pmatrix}$ is $\{\begin{pmatrix}a\\0\\b\end{pmatrix} : a, b \in \mathbb{R}\}$

Geometrically, the span is



xz-plane

Definition 6: a set of vectors $v_1, \dots, v_m$ is a spanning set for $\mathbb{R}^n$ iff every vector $w \in \mathbb{R}^n$ is given by a linear combination of $v_1, \dots, v_m$.

Example 6: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Is there anything weird in Example 6?

In general, we will want to avoid "redundancies" in our spanning sets. Here, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ can be obtained as the sum of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

Definition 7: A set of vectors $v_1, \dots, v_m$ is linearly independent iff whenever $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m = 0$, we must have $\lambda_1 = \lambda_2 = \dots = \lambda_m$.

In other words, the only linear combination summing to 0 is the obvious one: when all the λ_i are 0.

Example 7: $\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ are linearly independent: if $\lambda \begin{pmatrix}1\\0\\2\end{pmatrix} + \mu \begin{pmatrix}0\\1\\0\end{pmatrix} = 0$, then $\begin{pmatrix}\lambda\\ \mu\\ 2\lambda\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, so $\lambda = \mu = 0$.

Example 8: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ are not linearly independent: $1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} - 1 \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Discussion: we have two potential notions of "no redundancies":
- No linear dependence
- Minimum amount of gens

The following theorem says that these two notions are actually the same.

Theorem 1: A set of vectors $v_1, \dots, v_m$ is a "minimal" spanning set for $\mathbb{R}^n$ if and only if $v_1, \dots, v_m$ is a "maximal" linearly independent set in $\mathbb{R}^n$.

"minimal" spanning set: it is spanning and if you remove any vector, it stops being spanning

"maximal" linearly independent set: it is linearly independent and if you add another vector, it stops being lin. indep.

Definition 8: A basis for $\mathbb{R}^n$ is a spanning set which is also linearly independent.

Example 9: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ form a basis. This is called the "canonical basis" of $\mathbb{R}^n$.

Proof of Theorem 1: (For the mathematically inclined)

$\Rightarrow$) Assume $v_1, \ldots, v_m$ is a minimal spanning set. We prove first that it is linearly independent.

So set a linear combination $\lambda_1 v_1 + \ldots + \lambda_m v_m$ equal to zero and let us prove that $\lambda_1 = \ldots = \lambda_m = 0$.

Suppose that is not the case, and one of the λ_i's is nonzero. After reordering the λ_i's, we may assume $\lambda_1 \neq 0$. But then, $v_1 = \dfrac{-\lambda_2 v_2 - \ldots - \lambda_m v_m}{\lambda_1}$, and so the set $v_2, \ldots, v_m$ is also a generating set.

This contradicts the minimality of $v_1, \ldots, v_m$. Thus our assumption was wrong and therefore $\lambda_1 = \ldots = \lambda_m = 0$.

Next, let us prove that the set is maximally linearly independent. So add a new vector v_{m+1}.

Since $v_1, \ldots, v_m$ is a spanning set, $v_{m+1} = \lambda_1 v_1 + \ldots + \lambda_m v_m$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, not all of them zero.

But then $\lambda_1 v_1 + \ldots + \lambda_m v_m - v_{m+1} = 0$, so $v_1, \ldots, v_{m+1}$ is no longer linearly independent.

$\Leftarrow$) Assume $v_1, \ldots, v_m$ is maximally linearly independent. We show first that $v_1, \ldots, v_m$ is spanning.

Assume $v_1, \ldots, v_m$ did not span some vector v_{m+1}. Then $v_1, v_2, \ldots, v_m, v_{m+1}$ would be linearly independent:

if $\lambda_1 v_1 + \ldots + \lambda_m v_m + \lambda_{m+1} v_{m+1} = 0$, then we have two cases:

- $\lambda_{m+1} \neq 0$. Then $v_{m+1} = \dfrac{-\lambda_1 v_1 - \ldots - \lambda_m v_m}{\lambda_{m+1}}$, contradicting the assumption that $v_1, \ldots, v_m$ do not generate v_{m+1}.
- $\lambda_{m+1} = 0$. Then $\lambda_1 v_1 + \ldots + \lambda_m v_m = 0$, which implies $\lambda_1 = \ldots = \lambda_m = 0$ since $v_1, \ldots, v_m$ are linearly independent.

We conclude that $\lambda_1 = \ldots = \lambda_m = \lambda_{m+1} = 0$, so $v_1, \ldots, v_m, v_{m+1}$ are linearly independent, contradicting the maximality assumption.

Finally we prove that the set is minimal spanning. Indeed, if $v_1, \ldots, v_{m-1}$ are spanning, then $v_m = \lambda_1 v_1 + \ldots + \lambda_{m-1} v_{m-1}$, so $\lambda_1 v_1 + \ldots + \lambda_{m-1} v_{m-1} - v_m = 0$. Thus $v_1, \ldots, v_m$ are not linearly independent, a contradiction. It follows that $v_1, \ldots, v_m$ are minimal spanning.

$\square$

In-class exercise session:

1. Write the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (one example is enough).
2. Do the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ span $\mathbb{R}^3$? Why or why not?
3. Find a linearly indep spanning set for the set of vectors $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a+b=c \right\}$.