Lecture 17

Solutions to in-class exercises:

1. Find the QR factorization of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$.

Gram-Schmidt: $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_2^{\perp} = v_2 - (u_1 \cdot v_2)\, u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{-2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}}_{R}$$

2. Let $S = \mathrm{Im}(A)$. Find the matrix of proj_S: matrix of proj_S is $QQ^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$

$$= \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}$$

3. $\|T(v)\|^2 = T(v) \cdot T(v) = v \cdot v = \|v\|^2 \xRightarrow{\text{Both are } \geq 0} \|T(v)\| = \|v\|$

$((AB)^T)_{ij} = (j\text{th row of } A)^T \cdot (i\text{th col of } B)$, and $(B^T A^T)_{ij} = (i\text{th row of } B^T)^T \cdot (j\text{th col of } A^T)$, so $(AB)^T = B^T A^T$.

Recap: • A linear transformation with $n \times n$ matrix Q is orthogonal iff $Q = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$ orthonormal basis

↳ Equivalently, $Q^T Q = I_n$ (i.e. $Q^T = Q^{-1}$).

↳ Equivalently, T preserves the dot product: $T(v) \cdot T(w) = v \cdot w$

Today: more on orthogonality + orthogonal diagonalization.

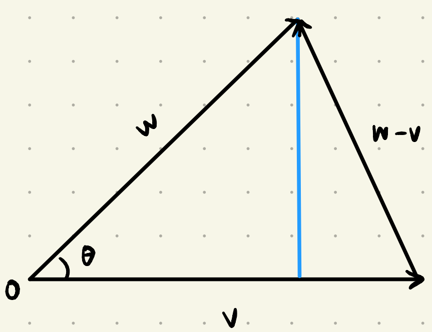
First: <u>how to picture orthogonal transformations</u>

What does it mean geometrically that T preserves the dot product? We have seen it preserves length.

Now we'll see: if T is orthogonal, then it preserves angles.

Question: what's the relation between the dot product of v and w and their dot product?

Answer: if we put v and w on the plane, and v on the x-axis of $\mathbb{R}^2$, we get:



Notice first: $\|w-v\|^2 = (w-v)\cdot(w-v) = v\cdot v + w\cdot w - 2v\cdot w$

$$= \|v\|^2 + \|w\|^2 - 2v\cdot w \qquad (*)$$

On the other hand, the coordinates of w are, by trigonometry, $(\|w\|\cos\theta, \|w\|\sin\theta)$.

the coordinates of v are clearly $(\|v\|, 0)$

So $\|w-v\| = \|(\|w\|\cos\theta - \|v\|, \|w\|\sin\theta)\|$

$$= (\|w\|\cos\theta - \|v\|)^2 + \|w\|^2\sin^2\theta$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \|v\|^2 + \|w\|^2 - 2\|v\|\cdot\|w\|\cos\theta$$

Comparing this with $(*)$, we get $v\cdot w = \|v\|\cdot\|w\|\cos\theta$

Upshot: dot product encapsulates lengths + angles, so if T is orthogonal,

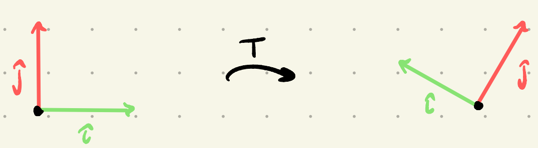
$$\cos(\text{angle}(T(v),T(w))) = \frac{T(v)\cdot T(w)}{\|T(v)\|\cdot\|T(w)\|} = \frac{v\cdot w}{\|v\|\cdot\|w\|} = \cos(\text{angle}(v,w)).$$

2x2 orthogonal matrices

Let Q be a 2×2 orthogonal matrix. Then the columns of Q are unit vectors at a $90°$ angle, so

rotation by θ, $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

OR



rotation with vectors swapped, $Q = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

Observe: these preserve lengths and angles.

Remark: the orthogonal 3×3 matrices of determinant 1 represent all possible rotations in 3D-space. They form a group called $SO(3)$, which has important applications in physics and engineering.

Orthogonal diagonalization

Recall: diagonalizing $T: \mathbb{R}^n \to \mathbb{R}^n$ is finding a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal.

Definition 1: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then orthogonally diagonalizing T is finding an <u>orthonormal</u> basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal.

Example 1: let $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$. Eigenvalues: char poly$(A) = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

$$E_4 = \operatorname{Ker}\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \Rightarrow \text{Orthonormal basis: } u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$E_{-1} = \operatorname{Ker}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) \Rightarrow \text{Orthonormal basis: } u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

"Luckily", u_1 and u_2 are orthogonal: $u_1 \cdot u_2 = \frac{1}{5}(2 \cdot 1 + 1 \cdot (-2)) = 0 \Rightarrow u_1, u_2$ are an orthonormal basis of eigenvectors.

Bonus: Easy to diagonalize! let $Q = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$. Then

$$A = Q\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}Q^{-1} = Q\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}Q^T$$

$$= \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

Orthogonal diagonalization of A

Recall: A is diagonalizable $\Leftrightarrow \sum g_\lambda = n$.

Obvious question: when is A orthogonally diagonalizable?

Observe that if $A = \underset{\text{orthog.}}{Q}\,\underset{\text{diagonal}}{D}\,Q^T$, then $A^T = QDQ^T = A$.

Definition 2: An $n \times n$ matrix is symmetric iff $A^T = A$.

So, orthogonally diagonalizable matrices are definitely symmetric. Now, which symmetric matrices are orthogonally diagonalizable?

Answer: amazingly, all of them.

Theorem 1 (Spectral theorem): Let A be an $n\times n$ symmetric matrix. Then

1) The eigenvalues of A are real and $\sum_{\lambda \text{ eig. of } A} a_\lambda = n$
2) The eigenspaces of A are mutually orthogonal (if $v\in E_\lambda$, $w\in E_\mu$, $\lambda\neq\mu$, then $v\cdot w = 0$).
3) A is orthogonally diagonalizable.

Proof: 1) Consider the eigenvalues of A over $\mathbb{C}$, $\lambda_1,\dots,\lambda_k$. By the "fundamental" theorem of algebra, $a_{\lambda_1}+\dots+a_{\lambda_k} = n$. So it suffices to show that $\lambda_1,\dots,\lambda_k$ are actually real. So take one of the eigenvalues $\lambda = a+bi$. We want to show that $b=0$.
Now, take an eigenvector $v+iw\in E_\lambda$. Then

$$\left.\begin{aligned}
&(v+iw)^T \underbrace{A(v-iw)}_{(a-bi)(v-iw)} = (a-bi)(v+iw)^T(v-iw)\\
&\underbrace{(v+iw)^T A}_{(A^T(v+iw))^T \;\underset{A \text{ symmetric}}{=}\; (\underbrace{A(v+iw)}_{(a+bi)(v+iw)})^T}(v-iw) = (a+bi)(v+iw)^T(v-iw)
\end{aligned}\right\} \Rightarrow a-bi = a+bi \quad \text{i.e. } b=0.$$

2) Similar trick: if $v\in E_\lambda$ and $w\in E_\mu$,

$$\left.\begin{aligned}
v^TAw &= v^T(Aw) = \mu v^Tw\\
v^TAw &= (Av)^Tw = \lambda v^Tw
\end{aligned}\right\} \text{ since } \lambda\neq\mu,\ v^Tw = 0$$

3) Proof for $n=2$: take an eigenvector v and complete it to an orthonormal basis $\mathcal{B}$: u_1, u_2.
Then $A = Q^T\begin{pmatrix}\lambda & * \\ 0 & *\end{pmatrix}Q$. Now notice that $\begin{pmatrix}\lambda & * \\ 0 & *\end{pmatrix} = QAQ^T$, and the RHS is symmetric: $(QAQ^T)^T = QA^TQ^T = QAQ^T$. It follows that $\begin{pmatrix}\lambda & * \\ 0 & *\end{pmatrix} = \begin{pmatrix}\lambda & 0 \\ 0 & \mu\end{pmatrix}$, as we wanted.

Proof for $n=3$: take an eigenvector and complete it to an orthonormal basis u_1, u_2, u_3.
Then $A = Q^T\begin{pmatrix}\lambda & * & * \\ 0 & * & * \\ 0 & * & *\end{pmatrix}Q$. Same argument $\Rightarrow$ $A = \begin{pmatrix}\lambda & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{B}} \\ 0 & & \end{pmatrix}$. Now by the previous case, $B = Q_0^TDQ_0$, so $A = Q^TQ_0^T\begin{pmatrix}\lambda & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2\end{pmatrix}Q_0Q$, as desired $\square$.

Algorithm for orthogonal diagonalization

1. Is A symmetric? If not, A is <u>not</u> orthogonally diagonalizable.
2. Find an eigenbasis (always possible).
3. Find an orthonormal basis for each eigenspace.

Example 2: Orthogonally diagonalize the matrix $A = \begin{pmatrix} 5/4 & 0 & \sqrt{3}/4 \\ 0 & 1 & 0 \\ \sqrt{3}/4 & 0 & 7/4 \end{pmatrix}$.

$$\text{char poly}(A) = \det \begin{pmatrix} 5/4-\lambda & 0 & \sqrt{3}/4 \\ 0 & 1-\lambda & 0 \\ \sqrt{3}/4 & 0 & 7/4-\lambda \end{pmatrix}$$

$$= (1-\lambda)\cdot \det \begin{pmatrix} 5/4-\lambda & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4-\lambda \end{pmatrix} = (1-\lambda)\cdot\left(\lambda^2 - 3\lambda + \frac{35}{16} - \frac{3}{16}\right) = (1-\lambda)^2(2-\lambda)$$

$$E_2 = \operatorname{Ker}(A - 2I_3) = \operatorname{Ker}\begin{pmatrix} -3/4 & 0 & \sqrt{3}/4 \\ 0 & -1 & 0 \\ \sqrt{3}/4 & 0 & -1/4 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \end{pmatrix} \longrightarrow \text{Orthonormal basis: } \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \end{pmatrix}$$

$$E_1 = \operatorname{Ker}(A - I_3) = \operatorname{Ker}\begin{pmatrix} 1/4 & 0 & \sqrt{3}/4 \\ 0 & 0 & 0 \\ \sqrt{3}/4 & 0 & 3/4 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ -\sqrt{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

Next, Gram-Schmidt for the basis $v_1 = \begin{pmatrix} 1 \\ 1 \\ -\sqrt{3} \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ of E_1:

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 1 \\ -\sqrt{3} \end{pmatrix}, \quad v_2^\perp = v_2 - (u_1 \cdot v_2)u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ -\sqrt{3} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 \\ 4 \\ \sqrt{3} \end{pmatrix}, \quad u_2 = \frac{1}{2\sqrt{5}}\begin{pmatrix} -1 \\ 4 \\ \sqrt{3} \end{pmatrix}.$$

Finally,

$$A = \begin{pmatrix} 1/2 & 1/\sqrt{5} & -1/2\sqrt{5} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1/\sqrt{3} & -\sqrt{3}/\sqrt{5} & \sqrt{3}/2\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/\sqrt{3} \\ 1/\sqrt{5} & 1/\sqrt{5} & -\sqrt{3}/\sqrt{5} \\ -1/2\sqrt{5} & 2/\sqrt{5} & \sqrt{3}/2\sqrt{5} \end{pmatrix}$$

In-class exercises:

1. Prove that an orthogonal matrix must have determinant 1 or -1.
2. Find the orthogonal diagonalization of $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$