Lecture 15

In-class exercise from last time: diagonalize the matrix $A=\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$ over $\mathbb{C}$.

$$\text{char poly}(A) = \det\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3+1$$

Roots of $-\lambda^3+1$: $\lambda=1$ works since $-1^3+1=0$. So divide $-\lambda^3+1$ by $\lambda-1$ $\rightsquigarrow$ $-\lambda^3+1 = -(\lambda-1)\cdot(\lambda^2+\lambda+1)$ (Use long division)

Now the roots of $\lambda^2+\lambda+1$ are $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$.

Eigenspaces: $E_1 = \operatorname{Ker}\begin{pmatrix} -1 & & 1 \\ 1 & -1 & \\ & 1 & -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{I \to -I} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{II \to II - I} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{II \to -II} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{III \to III - II} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$\hookrightarrow t$

$$\Rightarrow E_1 = \operatorname{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$E_{\frac{-1+i\sqrt{3}}{2}} = \operatorname{Ker}\begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0 & 1 \\ 1 & \frac{1-i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} \end{pmatrix}$$

$$\left(\frac{1-i\sqrt{3}}{2}\right)^{-1} = \frac{2}{1-i\sqrt{3}} = \frac{2(1+i\sqrt{3})}{4}$$

$$\left(\begin{array}{ccc|c} \frac{1-i\sqrt{3}}{2} & 0 & 1 & 0 \\ 1 & \frac{1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} & 0 \end{array}\right) \xrightarrow{I \to \frac{1}{\frac{1-i\sqrt{3}}{2}} I} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 1 & \frac{1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} & 0 \end{array}\right)$$

$$\xrightarrow{\mathrm{II}\to\mathrm{II}-\mathrm{I}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} & 0 \end{array}\right)$$

$$\xrightarrow{\mathrm{II}\leftrightarrow\mathrm{III}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 0 \end{array}\right)$$

$$\left(\frac{1-i\sqrt{3}}{2}\right)^2 = \frac{1-3-2i\sqrt{3}}{4} = \frac{-1-i\sqrt{3}}{2}$$

$$\xrightarrow{\mathrm{III}\to\mathrm{III}-\frac{1-i\sqrt{3}}{2}\mathrm{II}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1-i\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow E_{\frac{-1+i\sqrt{3}}{2}} = \mathrm{Span}\left(\begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix}\right)$$

$$E_{\frac{-1-i\sqrt{3}}{2}} = \mathrm{Ker}\begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 0 & 1 \\ 1 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1+i\sqrt{3}}{2} \end{pmatrix}$$

Useful observation: let $\bar{A} = (\bar{a}_{ij})_{i,j=1,\dots,n}$, and $\bar{v} = \begin{pmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{pmatrix}$.

If $Av = 0$, then $\bar{A}\bar{v} = \overline{Av} = \bar{0} = 0$

In other words, $v \in \mathrm{Ker}(A) \Leftrightarrow \bar{v} \in \mathrm{Ker}(\bar{A})$

Therefore $E_{\frac{-1-i\sqrt{3}}{2}} = \mathrm{Span}\left(\begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix}\right)$

Finally, $S = \begin{pmatrix} 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}$

$$S^{-1}: \left(\begin{array}{ccc|ccc} 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1 & 0 & 0 \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{III}\to\mathrm{III}-\mathrm{I}]{\mathrm{II}\to\mathrm{II}-\mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1 & 0 & 0 \\ 0 & i\sqrt{3} & -i\sqrt{3} & -1 & 1 & 0 \\ 0 & \frac{3+i\sqrt{3}}{2} & \frac{3-i\sqrt{3}}{2} & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{II}\to\frac{1}{i\sqrt{3}}=\frac{-i\sqrt{3}}{3}} \left(\begin{array}{ccc|ccc} 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{+i\sqrt{3}}{3} & \frac{-i\sqrt{3}}{3} & 0 \\ 0 & \frac{3+i\sqrt{3}}{2} & \frac{3-i\sqrt{3}}{2} & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\mathrm{III}\to\mathrm{III}-\frac{3+i\sqrt3}{2}\mathrm{II}]{\mathrm{I}\to\mathrm{I}+\frac{1+i\sqrt3}{2}\mathrm{II}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3-i\sqrt3}{6} & \frac{3+i\sqrt3}{6} & 0 \\ 0 & 1 & -1 & \frac{+i\sqrt3}{3} & \frac{-i\sqrt3}{3} & 0 \\ 0 & 0 & 3 & \frac{-1-i\sqrt3}{2} & \frac{-1+i\sqrt3}{2} & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{III}\to\frac{1}{3}\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3-i\sqrt3}{6} & \frac{3+i\sqrt3}{6} & 0 \\ 0 & 1 & -1 & \frac{+i\sqrt3}{3} & \frac{-i\sqrt3}{3} & 0 \\ 0 & 0 & 1 & \frac{-1-i\sqrt3}{6} & \frac{-1+i\sqrt3}{6} & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[\mathrm{II}\to\mathrm{II}+\mathrm{III}]{\mathrm{I}\to\mathrm{I}+\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1+i\sqrt3}{6} & \frac{-1-i\sqrt3}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{-1-i\sqrt3}{6} & \frac{-1+i\sqrt3}{6} & \frac{1}{3} \end{array}\right)$$

So
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-1-i\sqrt3}{2} & \frac{-1+i\sqrt3}{2} \\ 1 & \frac{-1+i\sqrt3}{2} & \frac{-1-i\sqrt3}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac{-1-i\sqrt3}{2} & \\ & & \frac{-1+i\sqrt3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1+i\sqrt3}{6} & \frac{-1-i\sqrt3}{6} & \frac{1}{3} \\ \frac{-1-i\sqrt3}{6} & \frac{-1+i\sqrt3}{6} & \frac{1}{3} \end{pmatrix}.$$

Finishing up: Assorted questions

5. The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.

False: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

6. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then there must be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

True (By definition)

7. If the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are eigenvectors of an $n \times n$ matrix A, then A must be diagonal.

True: $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where λ_i is the eig. of e_i

8. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^3 as well.

True: $A^3 v = \lambda^3 v$

9. There exists a diagonalizable 5×5 matrix with only two distinct eigenvalues (over $\mathbb{C}$).

10. There exists a real 5×5 matrix without any real eigenvalues.

False: char poly has deg 5, say $p(\lambda)$. Then $\lim_{\lambda\to\infty} p(\lambda)$ and $\lim_{\lambda\to-\infty} p(\lambda)$ are $\pm\infty$ $\Rightarrow$ There is some $\lambda \in \mathbb{R}$ s.t. $p(\lambda) = 0$.

17. If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.

False: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

18. If A and B are $n \times n$ matrices, if α is an eigenvalue of A, and if β is an eigenvalue of B, then $\alpha\beta$ must be an eigenvalue of AB.

False: $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

A: eigs $\pm\sqrt2$; B: eigs ±1; AB: eigs $2, 1$

19. If 3 is an eigenvalue of an $n \times n$ matrix A, then 9 must be an eigenvalue of A^2.

True: $Av = 3v$
$\Rightarrow A^2 v = 9v$

20. The matrix of any orthogonal projection onto a subspace V of $\mathbb{R}^n$ is diagonalizable.

21. All diagonalizable matrices are invertible.

False: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

22. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ must be an eigenvector of $A + B$.

True: $(A+B)v = Av + Bv = \lambda v + \mu v = (\lambda+\mu)v$

23. If matrix A^2 is diagonalizable, then matrix A must be diagonalizable as well.

False: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonalizable, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in JNF and is not diagonal

24. The determinant of a matrix is the product of its eigenvalues (over $\mathbb{C}$), counted with their algebraic multiplicities.

25. All lower triangular matrices are diagonalizable (over $\mathbb{C}$).

False: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is not diagonalizable: $a_1 = 2$ whereas $g_1 = \dim(E_1)$

$= \dim\left(\ker\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$

$= 2 - \operatorname{rank}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1$

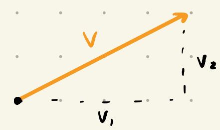
Today and next week: orthogonality

Idea: length and angles

Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Then one can define the length of v as $\sqrt{v_1^2 + \ldots + v_n^2}$.

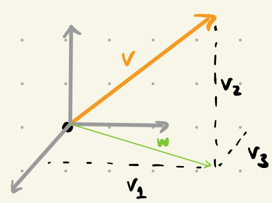
This matches our intuition:

- In $\mathbb{R}^2$,



$\Rightarrow \text{length}(v) = \sqrt{v_1^2 + v_2^2}$ (Pythagorean theorem)

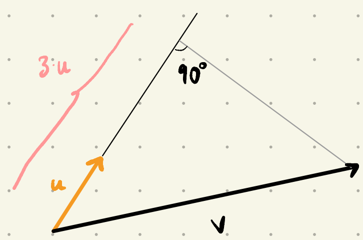
- In $\mathbb{R}^3$,



Pythagoras: $\text{length}(v) = \sqrt{v_2^2 + \text{length}(v)^2}$

$= \sqrt{v_1^2 + v_2^2 + v_3^2}$

Next, fix a vector $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ of length 1. One can consider the transformation $T: \mathbb{R}^n \to \mathbb{R}$ given by the matrix $(u_1 \cdots u_n)$. This represents an "orthogonal projection" onto the line spanned by u:



$\Rightarrow T(v) = 3$

(This is explained in 3Blue1Brown's video)

Observe that u and v are "perpendicular" iff $T(v) = 0$ i.e. iff $u^T v = 0$.

Definition 1: • Let $v, w \in \mathbb{R}^n$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. Then the dot product of v and w is

$$v \cdot w = v^T w = v_1 w_1 + \ldots + v_n w_n.$$

- The length of v is $\|v\| = \sqrt{v \cdot v}$. If $\|v\| = 1$, then v is a unit vector.
- The vectors v and w are orthogonal iff $v \cdot w = 0$.

Let S be a subspace of $\mathbb{R}^n$. In what follows, we will be interested in finding bases of S consisting of unit vectors, all pairwise orthogonal.

Definition 2: Let $u_1, \dots, u_k \in \mathbb{R}^n$. Then $u_1, \dots, u_k$ are orthonormal iff $\|v_i\| = 1$ for all i and

$u_i \cdot u_j = 0$ for all i, j s.t. $i \neq j$. (More succintly: $v_i \cdot v_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$)

Example 1: the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are orthonormal.

Theorem 1: Orthonormal vectors are linearly independent.

Proof: Suppose $u_1, \dots, u_k$ are orthonormal. Take a linear dependence $\lambda_1 u_1 + \dots + \lambda_k u_k$. We want to show that $\lambda_1 = \dots = \lambda_k = 0$.

Now $0 = u_i \cdot (0) = u_i \cdot (\lambda_1 u_1 + \dots + \lambda_k u_k) \underset{u_i \cdot u_j = \{^1_0}{=} \lambda_i$ for all i, as desired. ◻

Corollary: Let $S \subseteq \mathbb{R}^n$ be a subspace of dimension s. Then, any set of s orthonormal vectors in S forms a basis of S.

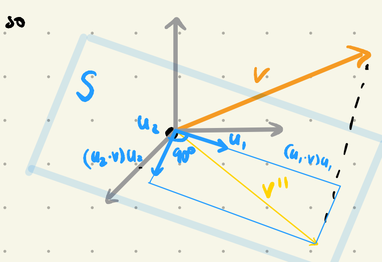
Orthogonal projections

Fix a subspace $S \subseteq \mathbb{R}^n$, and assume $u_1, \dots, u_s$ is an orthonormal basis of S.

Definition 3: Let $v \in \mathbb{R}^n$. Then the orthogonal projection of v onto S is given by:

$$v^{\parallel} = (u_1 \cdot v) u_1 + (u_2 \cdot v) u_2 + \dots + (u_s \cdot v) u_s$$

Picture: $u_i \cdot v$ is the "coordinate in the direction of u_i", so



Definition 4: The perpendicular component of v is $v^{\perp} = v - v^{\parallel}$.

Theorem 2: The vector $v^{\perp}$ is orthogonal to the subspace S (i.e. orthogonal to every vector in S).

Proof: First note that $u_i \cdot v^{\perp} = u_i \cdot (v - v^{\parallel})$

$$= (u_i \cdot v) - u_i \cdot \left((u_1 \cdot v) u_1 + (u_2 \cdot v) u_2 + \dots + (u_s \cdot v) u_s \right)$$

$$= (u_i \cdot v) - (u_i \cdot v) \cdot \underbrace{(u_i \cdot u_i)}_{1} = 0.$$

Now, if $w \in S$, we can write it as $w = \lambda_1 u_1 + \dots + \lambda_s u_s$, so $v^{\perp} \cdot w = \lambda_1 \underbrace{(v^{\perp} \cdot u_1)}_{0} + \dots + \lambda_s \underbrace{(v^{\perp} \cdot u_s)}_{0}$

$= 0$. $\square$

Example 2: Consider $S = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$, and $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Note that $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form an orthonormal basis.

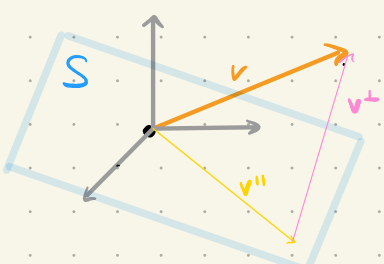
Then $v^{\parallel} = (v \cdot u_1) \cdot u_1 + (v \cdot u_2) \cdot u_2 = \frac{3}{\sqrt{2}} \cdot u_1 + 3 \cdot u_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ 3 \end{pmatrix}$

$v^{\perp} = v - v^{\parallel} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$.

Observe: $v^{\perp} \cdot u_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

$v^{\perp} \cdot u_2 = 0 + 0 = 0$

Remark: observe that every $v \in \mathbb{R}^n$ can be written uniquely as $v = v^{\parallel} + v^{\perp}$.



Important remark: $v^{\parallel}$ and $v^{\perp}$ are <u>always</u> in reference to some subspace $S \subseteq \mathbb{R}^n$. Sometimes, we write $v^{\parallel} = \text{proj}_S(v)$ every time, to be less ambiguous.

Observation: If $S = \mathbb{R}^n$, $\text{proj}_S =$ identity map, so if $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$, every $v \in \mathbb{R}^n$ can be written as $v = (u_1 \cdot v) u_1 + \dots + (u_n \cdot v) u_n$.

Definition 5: The orthogonal complement of a subspace $S \subseteq \mathbb{R}^n$ is $S^{\perp} = \{ v \in \mathbb{R}^n : v \text{ is orthogonal to } S \}$

$= \{ v \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } w \in S \}$

Theorem 3: Let $S \subseteq \mathbb{R}^n$ be a linear subspace and let $u_1, \dots, u_s$ be an orthonormal basis of S. Then the transformation $\text{proj}_S : \mathbb{R}^n \to \mathbb{R}^n$ is linear, its image is S and its kernel is $S^{\perp}$.

Proof: To see that proj_S is linear, we need to show:

- $\text{proj}_S(v+w) = \text{proj}_S(v) + \text{proj}_S(w)$ (In-class exercise)

- $\text{proj}_S(\lambda v) = (u_1 \cdot \lambda v)u_1 + \dots + (u_n \cdot \lambda v)u_n$

$$\underset{\substack{\text{dot product} \\ \text{is linear}}}{=} \lambda (u_1 \cdot v)u_1 + \dots + \lambda (u_n \cdot v) u_n$$

$$= \lambda \left((u_1 \cdot v) u_1 + \dots + (u_n \cdot v) u_n \right)$$

$$= \lambda \, \text{proj}_S(v).$$

$\text{Im}(S) = S$ (In-class exercise).

$$\text{Ker}(S) = \{ v \in \mathbb{R}^n : (u_1 \cdot v) u_1 + \dots + (u_n \cdot v) u_n = 0 \}$$

$$\underset{u_1, \dots, u_n \text{ l.i.}}{=} \{ v \in \mathbb{R}^n : u_1 \cdot v = u_2 \cdot v = \dots = u_n \cdot v = 0 \}$$

$$\underset{u_1, \dots, u_n \text{ span } S}{=} \{ v \in \mathbb{R}^n : w \cdot v = 0 \text{ for all } w \in S \}$$

$$= S^\perp. \qquad \square$$

Corollary: $S^\perp$ is a subspace of $\mathbb{R}^n$

Natural question: what is the matrix of proj_S?

Example 3: Consider $S = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$, and $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ again.

Then
$$\text{proj}_S \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (e_1 \cdot u_1) \cdot u_1 + (e_1 \cdot u_2) \cdot u_2$$
$$= \frac{1}{\sqrt{2}} \cdot u_1 + 0 \cdot u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{proj}_S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (e_2 \cdot u_1) \cdot u_1 + (e_2 \cdot u_2) \cdot u_2$$
$$= \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{proj} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (e_1 \cdot u_1) u_1 + (e_2 \cdot u_2) u_2$$
$$= 0 u_1 + u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

proj_S is given by the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next time: how to find orthonormal bases?

In-class exercises:

Let $S = \text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$. Let $v = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Find $v^{\parallel}$ and $v^{\perp}$. Verify that $v^{\perp}$ is orthogonal to S.