The following theorem should not be surprising at this point:

Theorem 1: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then the following statements are equivalent:

1) A is diagonalizable.

2) There exists a basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

3) The dimensions of the eigenspaces add up to n.

You may worry that the bases for E_λ and $E_{\lambda'}$ are linearly dependent.

The following theorem says this cannot happen.

Theorem 1: If A is an $n\times n$ matrix, $v, w \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq c_2$ satisfy $Av = c_1 v$ and $Aw = c_2 w$ then v and w are linearly independent.

<u>Proof</u>: Suppose $\lambda v + \mu w = 0$ $(*)$. Then $A(\lambda v + \mu w) = A \cdot 0 \Rightarrow \lambda c_1 v + \mu c_2 w = 0$

Multiplying $(*)$ by c_1, we get $c_1 \lambda v + c_1 \mu w = 0$

$\Big\}$ Subtracting these get $\mu(c_1 - c_2) w = 0$

Since $c_1 - c_2 \neq 0$, $\mu w = 0 \Rightarrow \mu = 0$. Similarly, $\lambda = 0$. Thus v and w are l.i. $\square$.

In-class exercises:

1. Determine whether the following matrices are diagonalizable and diagonalize them if possible:

1) $\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

2) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

3) $\begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}$

Lecture 13:

In-class exercises from last time:

1. Determine whether the following matrices are diagonalizable and diagonalize them if possible:

1) $\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

char poly: $\det \begin{pmatrix} -\lambda & -2 & & \\ 2 & -\lambda & & \\ & & -\lambda & -3 \\ & & 3 & -\lambda \end{pmatrix} = -\lambda \cdot \det \begin{pmatrix} -\lambda & & \\ & -\lambda & -3 \\ & 3 & -\lambda \end{pmatrix} - 2 \cdot \det \begin{pmatrix} -2 & & \\ & -\lambda & -3 \\ & 3 & -\lambda \end{pmatrix}$

$$= (\lambda^2 + 2) \cdot \det \begin{pmatrix} -\lambda & -3 \\ 3 & -\lambda \end{pmatrix} = (\lambda^2+2)(\lambda^2+3)$$

$\downarrow$

This has no roots

$\Rightarrow$ Not diagonalizable

2) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Char poly: $\det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (2-\lambda) \cdot \det \begin{pmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (2-\lambda)^2(-1-\lambda)$

Roots $\begin{cases} \lambda = 2 \\ \lambda = -1 \end{cases}$

Eigenspaces: $E_2 = \mathrm{Ker} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \rightarrow \dim(E_2) = 1$

$E_{-1} = \mathrm{Ker} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \rightarrow \dim(E_{-1}) = 1$

Since $\dim(E_2) + \dim(E_{-1}) = 2 < 3$, A is ~~not~~ not diagonalizable

3) $\begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}$

Char poly: $\det \begin{pmatrix} -\lambda & 1/2 & -1/2 \\ -4 & 3-\lambda & -4 \\ -3 & 3/2 & -5/2-\lambda \end{pmatrix} = -\lambda \cdot \det \begin{pmatrix} 3-\lambda & -4 \\ 3/2 & -5/2-\lambda \end{pmatrix} + 4 \cdot \det \begin{pmatrix} 1/2 & -1/2 \\ 3/2 & -5/2-\lambda \end{pmatrix} - 3 \cdot \det \begin{pmatrix} 1/2 & -1/2 \\ 3-\lambda & -4 \end{pmatrix}$

$$= -\lambda \cdot \left(\underbrace{(3-\lambda)(-5/2-\lambda)}_{\lambda^2 - \frac{1}{2}\lambda - \frac{15}{2}} + 6 \right) + 4 \cdot \left((-5/4 - \lambda/2) + 3/4 \right) - 3 \cdot \left(-2 + \frac{3}{2} - \frac{\lambda}{2} \right)$$

$$= -\lambda \cdot \left(\lambda^2 - \tfrac{1}{2}\lambda - \tfrac{3}{2}\right) + (-2\lambda - 2) + \left(\tfrac{3}{2} + \tfrac{3\lambda}{2}\right)$$

$$= -\lambda^3 + \tfrac{1}{2}\lambda^2 + \left(\tfrac{3}{2} - 2 + \tfrac{3}{2}\right)\lambda - \tfrac{1}{2}$$

$$= -\lambda^3 + \tfrac{1}{2}\lambda^2 + \lambda - \tfrac{1}{2}$$

$\lambda = 1$ is a root $\Rightarrow$ $-\lambda^3 + \frac{1}{2}\lambda^2 + \lambda - \frac{1}{2} = (\lambda - 1)\cdot \underbrace{\left(-\lambda^2 - \frac{1}{2}\lambda + \frac{1}{2}\right)}$

$$\lambda = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}}{-2} = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$$

So char poly is $-(\lambda - 1)(\lambda + 1)(\lambda - \frac{1}{2})$

Eigenspaces:

- $\lambda = 1$: $E_1 = \operatorname{Ker}\begin{pmatrix} -1 & 1/2 & -1/2 \\ -4 & 2 & -4 \\ -3 & 3/2 & -7/2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 1/2 & -1/2 & 0 \\ -4 & 2 & -4 & 0 \\ -3 & 3/2 & -7/2 & 0 \end{array}\right) \xrightarrow[\mathrm{III} \to \frac{1}{-3}\mathrm{III}]{\mathrm{I} \to -\mathrm{I},\ \mathrm{II} \to \frac{1}{-4}\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 0 \\ 1 & -1/2 & 1 & 0 \\ 1 & -1/2 & 7/6 & 0 \end{array}\right)$$

$$\xrightarrow{\mathrm{II} \to \mathrm{II} - \mathrm{I},\ \mathrm{III} \to \mathrm{III} - \mathrm{I}} \left(\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right)$$

$$\xrightarrow{\mathrm{II} \to 2\cdot\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{I} \to \mathrm{I} - 1/2\,\mathrm{II}]{\mathrm{III} \to \mathrm{III} - 2/3\,\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow \quad t$$

$$E_1 = \left\{ \begin{pmatrix} 1/2\, t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \right).$$

Check: $A \cdot \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ ✓

- $\lambda = -1$

$$E_{-1} = \operatorname{Ker}\begin{pmatrix} 1 & 1/2 & -1/2 \\ -4 & 4 & -4 \\ -3 & 3/2 & -3/2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ -4 & 4 & -4 & 0 \\ -3 & 3/2 & -3/2 & 0 \end{array}\right) \xrightarrow[\mathrm{III} \to \mathrm{III} + 3 \cdot \mathrm{I}]{\mathrm{II} \to \mathrm{II} + 4\mathrm{I}} \left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{III} \to \frac{1}{3}\mathrm{III}]{\mathrm{II} \to \frac{1}{6}\mathrm{II}} \left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{III} \to \mathrm{III} - \mathrm{II}]{\mathrm{I} \to \mathrm{I} - 1/2\,\mathrm{II}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow t$$

$$E_{-1} = \left\{ \begin{pmatrix} 0 \\ 1/2\,t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

Check: $\begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ ✓

- $\lambda = \frac{1}{2}$ $\quad E_{1/2} = \operatorname{Ker}\begin{pmatrix} -1/2 & 1/2 & -1/2 \\ -4 & 5/2 & -4 \\ -3 & 3/2 & -3 \end{pmatrix}$

$$\xrightarrow{\mathrm{I} \to -2 \cdot \mathrm{I}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -4 & 5/2 & -4 & 0 \\ -3 & 3/2 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{\substack{\mathrm{II} \to \mathrm{II} + 4\mathrm{I} \\ \mathrm{III} \to \mathrm{III} + 3 \cdot \mathrm{I}}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\mathrm{III} \to -\frac{2}{3}\mathrm{III}]{\mathrm{II} \to -\frac{2}{3}\mathrm{II}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\substack{\mathrm{I} \to \mathrm{I} + \mathrm{II} \\ \mathrm{III} \to \mathrm{III} - \mathrm{II}}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow E_{1/2} = \left\{ \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} \right\}$$

$$\downarrow t$$

$$= \operatorname{Span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

Check: $\begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$ ✓

Now $\dim(E_1) + \dim(E_{-1}) + \dim(E_{1/2}) = 3 = \dim(\mathbb{R}^3) \Rightarrow$ A is diagonalizable!

So let $\mathcal{B}$ be the basis consisting of $v_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$

Put $S = S_{\mathcal{B}\to\mathcal{E}} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1 & 1 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$. Then $S^{-1} =$

$$\left(\begin{array}{ccc|ccc} 1/2 & 0 & -1/2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{I}\to 2\mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{II}\to \mathrm{II}-\mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{III}\to \mathrm{III}-\mathrm{II}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1/2 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{III}\to -2\cdot\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & -2 \end{array}\right)$$

$$\xrightarrow[\mathrm{II}\to\mathrm{II}-\mathrm{III}]{\mathrm{I}\to\mathrm{I}+\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -2 \\ 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & -4 & 2 & -2 \end{array}\right)$$

Finally, $A = S_{\mathcal{B}\to\mathcal{E}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} S_{\mathcal{E}\to\mathcal{B}}$

$$= \begin{pmatrix} 1/2 & 0 & -1/2 \\ 1 & 1 & 0 \\ 0 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} -2 & 2 & -2 \\ 2 & -1 & 2 \\ -4 & 2 & -2 \end{pmatrix}$$

Recap: • A square, $\underset{v \neq 0}{Av = \lambda v}$ $\rightsquigarrow$ v is an eigenvector with eigenvalue λ

• Eigenspace: $E_\lambda = \operatorname{Ker}(A - \lambda I_n)$

• Diagonalizing A $\Leftrightarrow$ Finding an eigenbasis

$\Leftrightarrow$ Finding a basis of each E_λ and putting them together.

(Only possible if $\sum_{\lambda \text{ eig.}} \dim(E_\lambda) = n$).

Today: multiplicities.

Definition 1: Let A be an $n \times n$ matrix and let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Then the geometric multiplicity of λ is $g_\lambda = \dim(E_\lambda)$.

Example 1: In the last in-class exercise, $g_1 = 1$, $g_{-1} = 1$, $g_{1/2} = 1$.

Remark: A is diagonalizable iff $\sum_{\lambda \text{ eig}} g_\lambda = n$

Definition 2: Let A be an $n \times n$ matrix and let $\lambda \in \mathbb{R}$ be an eigenvalue of A. The algebraic multiplicity of λ is a_λ = # times λ appears as a root of the char poly of A.

Example 2: If char poly$(A) = (\lambda+2)^2(\lambda-3)^4(\lambda^2+\lambda+1)$, then $a_{-2} = 2$, $a_3 = 4$.

Obvious question: are these related? The answer is yes, as the following theorem shows.

Theorem 1: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $n \times n$ matrix A and let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Then, $g_\lambda \leq a_\lambda$.

Example 3: Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Then $g_2 = \dim(E_2) = \dim\left(\operatorname{Ker}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 2 - \operatorname{rank}\left(\operatorname{Ker}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1$

char poly$(A) = (2-\lambda)^2 \Rightarrow a_2 = 2$.

Proof of Theorem 1: Let $v_1, \dots, v_g$ be a basis for E_λ, and complete it to a basis

$\mathcal{B}: v_1, \dots, v_g, v_{g+1}, \dots, v_n$ of $\mathbb{R}^n$.

Changing bases, $[T]_{\mathcal{B}} = \begin{pmatrix} | & & | & & | \\ \underbrace{[T(v_1)]_{\mathcal{B}}}_{\substack{\text{lies in } E_\lambda \\ \Rightarrow \text{ l.c. of } v_1,\dots,v_g}} & \cdots & \underbrace{[T(v_g)]_{\mathcal{B}}}_{\substack{\text{lies in } E_\lambda \\ \Rightarrow \text{ l.c. of } v_1,\dots,v_g}} & \cdots & [T(v_n)]_{\mathcal{B}} \\ | & & | & & | \end{pmatrix}$

$$= \begin{array}{c} \\ g\{ \\ \\ \end{array}\left(\begin{array}{c|c} \begin{matrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{matrix} & * \\ \hline \underbrace{\quad 0 \quad}_{g} & C \end{array}\right) \leftarrow \text{Call this matrix } B$$

Fact: Since $A \sim B$, $\text{char poly}(A) = \text{char poly}(B)$ (see HW)

Finally, $\text{char poly}(B) = \det(B - t \cdot I_n) = \det \left(\begin{array}{c|c} \begin{matrix} \lambda-t & & 0 \\ & \ddots & \\ 0 & & \lambda-t \end{matrix} & * \\ \hline 0 & C - \lambda \cdot I_{n-g} \end{array}\right)$

$$\underset{\substack{\text{rref} \\ (\text{first } g \text{ columns})}}{=} (\lambda-t)^g \det \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ \hline 0 & C - \lambda \cdot I_{n-g} \end{array}\right)$$

$$\underset{\substack{\text{rref} \\ (\text{next columns})}}{=} (\lambda-t)^g \cdot \underbrace{\det(C - \lambda \cdot I_{n-g})}_{\text{char poly}(C)} \Rightarrow a_\lambda \geq g.$$

Theorem 2: If A is diagonalizable, then $a_\lambda = g_\lambda$ for all eigenvalues λ of A.

Proof: Note that $n = \deg(\text{char poly}(A)) \geq \sum a_\lambda \underset{\text{Theorem 1}}{\geq} \sum g_\lambda \underset{A \text{ diagonalizable}}{=} n$

Therefore every "$\geq$" is actually an equality, so $\sum a_\lambda = \sum g_\lambda$. Since $a_\lambda \geq g_\lambda$ for each λ, it follows that $a_\lambda = g_\lambda$ for each λ. $\square$

Example 4: Show that the matrix $A=\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 7 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$ is <u>not</u> diagonalizable.

$$\text{char poly}(A) = (2-\lambda)^2(3-\lambda)(4-\lambda)(5-\lambda) \implies a_2 = 2$$

Now $g_2 = \dim\left(\operatorname{Ker}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}\right) = 5 - \underbrace{\text{rank}}_{4} = 1 < 2 = a_\lambda \implies A$ is <u>not</u> diagonalizable

Theorem 3: If λ is an eigenvalue of an $n\times n$ matrix A, then $g_\lambda \geq 1$.

Proof: λ is an eigenvalue $\iff E_\lambda \neq \{0\} \iff E_\lambda$ has <u>some</u> nonzero vector $\iff g_\lambda = \dim(E_\lambda) \geq 1$.

Theorem 4: Let A be a square $n\times n$ matrix with n distinct eigenvalues. Then A is diagonalizable.

<u>Proof:</u> We want to show that $\sum_{\lambda \text{ eig.}} g_\lambda = n$. Now $n = \dim(\mathbb{R}^n) \geq \sum_{\lambda\text{ eig.}} g_\lambda \geq \sum_{\lambda \text{ eig.}} 1 = \#\text{eigenvalues} \underset{\text{by assumption}}{=} n$.

Thus we must have equalities throughout, so $\sum_{\lambda\text{ eig.}} g_\lambda = n$. $\square$

Example 5: $A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}$ has char poly $-(\lambda-\frac{1}{2})(\lambda+1)(\lambda-1)$. Since $\#$ eigenvalues $= 3 = n$,

A is diagonalizable.

Aside: Complex numbers

(Recommended reading: 7.5 in the textbook)

Complex numbers are of the form $a+bi$, with $a,b\in\mathbb{R}$, and i a number s.t. $i^2=-1$.

They can be added, subtracted, multiplied, and divided

$(a+bi)(c+di) = (ac-bd) + i(ad+bc)$

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\cdot\frac{b}{a^2+b^2}$$

In-class exercises:

1. Determine if the following matrices are diagonalizable.

a) $\begin{pmatrix} 1 & 7 & 8 & 9 & 10 & 11 \\ & 2 & 12 & 13 & 14 & 15 \\ & & 3 & 16 & 17 & 18 \\ & & & 4 & 19 & 20 \\ & 0 & & & 5 & 21 \\ & & & & & 6 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

2. a) Find a 5×5 matrix with eigenvalues -1 and 0, $a_{-1}=3$, $a_0=2$, $g_{-1}=1$, $g_0=2$.

b) Find a 7×7 matrix with eigenvalues -1 and 0, $a_{-1}=3$, $a_0=2$, $g_{-1}=1$, $g_0=2$.

3. Write $\frac{1}{2-3i}$ in the form $a+bi$.