Lecture 12	

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In-class exercises' solutions:
1 Let $A = \begin{pmatrix} -\frac{11}{2} & \frac{3}{2} \\ -18 & 5 \end{pmatrix}$ You're given that $A \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $A \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -4 \end{pmatrix}$
Compute A^{200} , rounding to 3 decimal places.
We change A to the bosis $\mathbf{B} \cdot \mathbf{v}_1 = \begin{pmatrix} 1 \\ \mathbf{s} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -\mathbf{y} \end{pmatrix}$. So set $\mathbf{S} = \begin{pmatrix} 1 & -1 \\ \mathbf{s} & -\mathbf{y} \end{pmatrix}$. Then $\mathbf{S}^{-1} = \begin{pmatrix} \mathbf{y} & -1 \\ \mathbf{s} & -\mathbf{y} \end{pmatrix}$.
and by last fectore, $A = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u & -1 \\ 3 & -1 \end{pmatrix}$ Now $A^{400} = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} (-1)^{100} & 0 \\ 0 & (\frac{1}{2})^{100} \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 3 & -4 \end{pmatrix}$
$\overline{T} \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$ to 3 dec. places
$= \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 12 & -3 \end{pmatrix}.$
2. If A~B, then there exists S invertible st. A = So BS
$B \sim C$ " S_1 $B = S_1 \subset S_2$ $R^* \xrightarrow{S_0} R^* \xrightarrow{S_1} R^*$
$S_{1} A = S_{1}^{-1}S_{1}C S_{1}S_{0}$ Now observe that $(S_{1}S_{0})^{-1} = S_{0}^{-1}S_{1}^{-1}S_{0} A \sim C$.
Crucial Recap
• Given a basis \mathcal{B} of \mathbb{R}^n with rectors v_2, \ldots, v_n , the \mathcal{B} -coordinates of $v \in \mathbb{R}^n$
are $[v]_{\mathbf{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, where $v = c_1 v_1 + \ldots + c_n v_n$.
• The canonical basis is $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and we denote it C.
• There exist change of basis matrices $S_{B \rightarrow e} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $S_{E \rightarrow B} = S_{B \rightarrow e}$
s.t. $[v]_e = S_{B \rightarrow e} [v]_B$ and $[v]_B = S_{e \rightarrow B} [v]_e$.
• Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ with matrix A , and a basis \mathcal{B} as above,
the matrix of T with respect to B is $S_{B \to e}^{-1} A S_{B \to e}$ (Then $[Tv]_B = S^{-1}AS[v]_B$) S=Se_e
• If there exists an invertible matrix $S s.t. A = S^{-1}BS$, then $A \sim B$.
• If $A \sim D$, where D is diagonal, then A is diagonalizable.

Definition 1. To diagonalize an nxn matrix is to find an invertible matrix S s.t. S-AS is diagonal
Discussion: In-class exercise revisited
Recall that the exercise gave us two vectors $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$. These had the
property that $Av_2 = -v_2$ and $Av_2 = \frac{1}{2}v_2$. This allowed us to compute A^{200} by hand!
Problem: in real life we're rarely handed such vectors. We have to find them
Definition 2: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a finear transformation with matrix A. Then a nonzero vector $v \in \mathbb{R}^n$
is an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$ if $Av = \lambda v$. Note: "eigen" means "own" in German.
Example 1: $\binom{1}{3}$ and $\binom{-1}{-4}$ are eigenvectors of $A = \begin{pmatrix} -\frac{M}{2} & \frac{5}{2} \\ -48 & 5 \end{pmatrix}$, with eigenvalues -1 and $\frac{1}{2}$ respectively.
Discussion: how to find these in the first place? Brilliant idea: if $Av = \lambda v$ then
Discussion: how to find these in the first place? Brilliant idea: if $Av = \lambda v$ then $(A - \lambda In)v = 0$ This is saying $A - \lambda In$ has a nonzero vector in its kernell
Therefore, $A - \lambda I_n$ is not invertible, so det $(A - \lambda I_n) = 0$!
So if we want to find λ , a good starting point is solving the equation (in λ):
$det(A - \lambda I_n) = 0$
Definition 3: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a finear transformation with matrix A. Then the characteristic
polynomial of T (or of A) is $det(A - \lambda I_n)$ (This is a polynomial in λ)
Example 1 (ctd'): Suppose we don't know the eigenvalues -1 and $\frac{1}{2}$ of A.
The equation $det(A - \lambda I_n) = 0$ says $det\left(\begin{pmatrix} -\frac{14}{2} & \frac{3}{2} \\ -18 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$
$\det\left(\begin{pmatrix}-\frac{14}{2}-\lambda & \frac{3}{2}\\-18 & 5-\lambda\end{pmatrix}\right) = 0$
$\left(-\frac{11}{2}-\lambda\right)\cdot\left(5-\lambda\right)+\frac{3}{2}\cdot13=0$
$\lambda^2 - 5\lambda + \frac{11}{2}\lambda - \frac{55}{2} + 27 = 0$ 225 = 4.5 ²
$\lambda^{2} + \frac{1}{2}\lambda - \frac{1}{2} = 0 \qquad \implies \qquad \lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \cdot (\frac{-1}{2})}}{2} = \left(\begin{array}{c} \lambda = \frac{1}{2} \\ \lambda = -1 \end{array} \right)$

• $\lambda = \frac{4}{2}$ Av $= \frac{1}{2}$ v \Leftrightarrow v \in Ker $(A - \frac{1}{2}T_{n})$. We know how to complet them! A $-\frac{1}{2}T_{n} = \left(-\frac{14}{2} - \frac{3}{2} - \frac{3}{2}\right) = \left(-\frac{6}{4} - \frac{3}{2}\right)$ $(A - \frac{1}{2}T_{n})$ v $= \binom{0}{0}$: $\left(-\frac{6}{4} - \frac{3}{2}\right) \left(\frac{0}{1+\frac{1}{2}T_{n}}\right)^{\frac{1}{2}+\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}+\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}}$ $\Rightarrow \left(\frac{1}{7}\right)$ v $= \binom{0}{0}$: $\left(-\frac{6}{4} - \frac{3}{2}\right) \left(\frac{0}{1+\frac{1}{2}T_{n}}\right)^{\frac{1}{2}+\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}}$ $\Rightarrow \left(\frac{1}{7}\right)$ v $= \binom{0}{0}$: $\left(-\frac{6}{4} - \frac{3}{2}\right) \left(\frac{0}{1+\frac{1}{2}}\right)^{\frac{1}{2}+\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}}$ $\Rightarrow \left(\frac{1}{7}\right)$ is on eigenvector of A with eigenvalue $\frac{1}{2}$. Notice: we can sock the base to estimate the det to estimate the des		t! We found									•
$ \begin{pmatrix} A - \frac{1}{2} I_{n} \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -4 & \frac{4}{2} \\ -4 & \frac{1}{2} \\ 0 \end{pmatrix} \stackrel{T - \frac{1}{4} I_{n}}{I_{n} - \frac{1}{4}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \stackrel{T - \frac{1}{4} I_{n}}{I_{n}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \stackrel{T - \frac{1}{4}}{O} \stackrel{T - \frac{1}{4}}{O} \stackrel{T - \frac{1}{4} I_{n}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \stackrel{T - \frac{1}{4}}{O} \stackrel{T - \frac{1}{4}}{O} \stackrel{T - \frac{1}{4} I_{n}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}} \stackrel{T - \frac{1}{4}} \stackrel{T - \frac{1}{4}} \stackrel{T - \frac{1}{4}}{I_{n} - \frac{1}{4}} \stackrel{T - \frac{1}{4}} T$	· ·	• $\lambda = \frac{1}{2}$	$Av = \frac{1}{2}$	V (=>	v E Ker	$(A - \frac{1}{2}I_n)$	We kno	w how to	compute these	· · · ·	•
$\Rightarrow \begin{pmatrix} \frac{1}{4} \\ \frac{1}{1} \end{pmatrix} \text{ is on eigenvector of A with eigenvalue } \frac{1}{2}.$ Notice we can such the basis to estimate $\frac{1}{4}$. Notice we can such the basis to estimate $\frac{1}{4}$. Notice we can such the basis to estimate $\frac{1}{4}$. $A = -4:$ $A - \lambda I_n = \begin{pmatrix} -\frac{11}{4} + 4 & \frac{3}{2} \\ -18 & 5 + 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{4} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}$ $Ker(A - \lambda I_n): \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ -18 & 6 \end{pmatrix} \stackrel{I \to \frac{2}{4}I}{I + \frac{1}{18}I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \stackrel{I \to \frac{1}{4}I}{I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $Ker(A - \lambda I_n) = \frac{1}{4} \begin{pmatrix} \frac{1}{5}t \\ t \end{pmatrix} t \in R, \frac{1}{7} = \text{Span}(\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}) Basis: \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$ $\Rightarrow \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \text{ is an eigenvector for A with eigenvalue -1.$ $\frac{1}{5} \text{ Solve the polynomial equation det } (A - \lambda I_n) = 0 \text{Get eigenvalues } \lambda_1, \dots, \lambda_n.$ $2 \text{ Find nonzero vectors in Ker(A - \lambda_2 I_n), \dots, Ker(A - \lambda_n I_n) \text{ forming a basis } B = v_{a_1}, \dots, v_n$ $(I Marning Step 2 may faill)$	· ·	A- <u>(</u> 1, -	$= \begin{pmatrix} -\frac{11}{2} - \frac{1}{2} & \frac{3}{2} \\ -18 & 5 \end{pmatrix}$	$\left(\begin{array}{c} 3\\ 2\\ -\frac{1}{2}\end{array}\right) = \left(\begin{array}{c} 1\\ -\frac{1}{2}\end{array}\right)$	$-6\frac{3}{2}$ $-18\frac{9}{2}$	· · ·					•
Notice: we can scale the bas to $e_{2} - 4 = 4$ $A - \lambda I_{n} = \begin{pmatrix} -\frac{11}{2} + 4 & \frac{3}{2} \\ -18 & 5 + 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix} \xrightarrow{I \to \frac{2}{18}I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{1}(A - \lambda I_{n}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 4 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ke_{2}(A - \lambda I_{n}) = \begin{pmatrix} 1 & \frac{1}{3} \\ t \end{pmatrix} + ter \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I = -\Gamma} \begin{pmatrix} 1 & -\frac$	••••	$\left(A - \frac{1}{2}I_n\right)$	$V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -6 & \frac{3}{2} \\ -8 & \frac{9}{2} \\ \end{pmatrix}$	$ \begin{array}{c} 0 \\ 1 \xrightarrow{1} \\ $	$ \begin{array}{c} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{array} $		$\frac{1}{4}$ $\begin{pmatrix} 10\\ 10 \end{pmatrix}$ $\begin{pmatrix} 1\\ 10 \end{pmatrix}$	-> Ker(A-1/j]	n)= { (+t): tel
• $\lambda = -4$: $A - \lambda I_n = \begin{pmatrix} -\frac{11}{2} + 1 & \frac{3}{2} \\ -18 & 5 + 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}$ $Ker(A - \lambda I_n): \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix} \xrightarrow{I \to -\frac{2}{13}I} \begin{pmatrix} 1 & -\frac{1}{3} \\ A & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I \to I} \begin{pmatrix} 1 & -\frac{1}{3} \\ -18 & 6 \end{pmatrix}$ $Ker(A - \lambda I_n): \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix} \xrightarrow{I \to -\frac{2}{13}I} \begin{pmatrix} 1 & -\frac{1}{3} \\ A & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I \to I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $Ker(A - \lambda I_n): = \frac{1}{4} \begin{pmatrix} \frac{1}{3}I \\ I \end{pmatrix} + \frac{1}{13}I \begin{pmatrix} 1 & -\frac{1}{3} \\ A & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I \to I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $Ker(A - \lambda I_n): = \frac{1}{4} \begin{pmatrix} \frac{1}{3}I \\ I \end{pmatrix} + \frac{1}{13}I \begin{pmatrix} 1 & -\frac{1}{3} \\ A & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I \to I} \begin{pmatrix} 1 & -\frac{1}{3} \\ A & -\frac{1}{3} \end{pmatrix}$ $\Rightarrow \begin{pmatrix} \frac{3}{4} \\ I \end{pmatrix}$ is an eigenvector for A with eigenvector -1 . Solve the polynomial equation det $(A - \lambda I_n) = 0 \xrightarrow{I \to I} $ Get eigenvecture $\lambda_1, \dots, \lambda_n$. 2. Find nonzero vectors in $Ker(A - \lambda_2 I_n), \dots, Ker(A - \lambda_K I_n)$ forming a bosis $B = v_a, \dots, v_n$. (I Marning: Step 2 may fail!)	• •	$\Rightarrow \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$	is on eige	envector of	A with e	sigenvalue 1	•••••				
• $\lambda = -4$: $A - \lambda I_n = \begin{pmatrix} -\frac{H}{2} + 4 & \frac{3}{2} \\ -18 & 5 + 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}$ $ker(A - \lambda I_n): \begin{pmatrix} -\frac{9}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix} \xrightarrow{I \to -\frac{2}{13}I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 4 & -\frac{1}{3} \end{pmatrix} \xrightarrow{I \to I - I} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ $ker(A - \lambda I_n) = \frac{1}{4} \begin{pmatrix} \frac{1}{3}t \\ t \end{pmatrix} + t \in R y = \operatorname{Span}(\begin{pmatrix} \frac{1}{3}t \\ 1 \end{pmatrix}) \longrightarrow \operatorname{Basis}: \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$ $\Rightarrow \begin{pmatrix} \frac{1}{3} \end{pmatrix}$ is an eigenvector for A with eigenvalue -1 . $\frac{hast}{I}: Algorithm$ to diagonalize a matrix: 1. Solve the polynomial equation $\det(A - \lambda I_n) = 0 \implies \operatorname{Get}$ eigenvalues $\lambda_1, \dots, \lambda_n$. 2. Find nonzero vectors in $\operatorname{Ker}(A - \lambda_2 I_n), \dots, \operatorname{Ker}(A - \lambda_n I_n)$ forming a basis $B = v_{a_1}, \dots, v_n$ (I Marning: Step 2 may fail!)	• •								Notice: we can to es.	scale the $\left(\frac{1}{7}\right) =$	basis (- ()
$\begin{aligned} & \operatorname{Ker}(A - \lambda I_{n}): \begin{pmatrix} -\frac{4}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}^{I \rightarrow -\frac{2}{3}I} \begin{bmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} \xrightarrow{I \rightarrow I - I} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix} \\ & \operatorname{Ker}(A - \lambda I_{n}) = \begin{cases} \frac{1}{3}I \\ \frac{1}{5} \end{bmatrix}^{I} + I \in \mathbb{R} Y = \operatorname{Span}(\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}) \longrightarrow \operatorname{Basis}: \begin{pmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \\ \Rightarrow \begin{pmatrix} \frac{1}{3} \end{pmatrix}^{I} \text{ is an eigenvector for } A \text{ with eigenvalue } -1. \\ & \\ & \\ & \\ & \\ & \\ & \\ & \\ & \\ & \\ $	• •	• $\lambda = -4$					· · ·				
$\begin{aligned} & \operatorname{Ker}(A - \lambda I_{n}): \begin{pmatrix} -\frac{4}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix}^{I \rightarrow -\frac{1}{4}I} \left(\begin{array}{c} 1 & -\frac{1}{3} \\ 1 & -\frac{1}{3} \end{array} \right) \xrightarrow{I \rightarrow II} \left(\begin{array}{c} 4 & -\frac{1}{3} \\ 0 & 0 \end{array} \right) \\ & \operatorname{Ker}(A - \lambda I_{n}) = \left\{ \begin{pmatrix} \frac{1}{3}t \\ t \end{pmatrix} + t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} \frac{1}{3}t \\ 1 \end{pmatrix} \right) Basis: \begin{pmatrix} \frac{1}{3} \end{pmatrix} \\ \\ 1 \end{pmatrix} \\ & \Rightarrow \begin{pmatrix} \frac{1}{3} \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } -1. \end{aligned}$ $\begin{aligned} & \operatorname{Algorithm}^{*} \text{ to diagonalize a matrix:} \\ & \operatorname{Solve fle polynomial equation } \det(A - \lambda I_{n}) = 0 \operatorname{Get eigenvalues } \lambda_{1}, \dots, \lambda_{n}. \\ & \operatorname{Rer}(A - \lambda_{n} I_{n}) \operatorname{Get eigenvalues } \lambda_{n}, \dots, \lambda_{n}. \\ & \operatorname{Rer}(A - \lambda_{n} I_{n}) \operatorname{Get eigenvalues } \lambda_{n}, \dots, \lambda_{n}. \\ & \operatorname{Rer}(A - \lambda_{n} I_{n}) \operatorname{Get eigenvalues } \lambda_{n}, \dots, \lambda_{n}. \\ & \operatorname{Rer}(A - \lambda_{n} I_{n}) \operatorname{Get eigenvalues } \lambda_{n}, \dots, \lambda_{n}. \\ & \operatorname{Rer}(A - \lambda_{n} I_{n}) \operatorname{Rer}(A - \lambda$	• •	4-21" =	$ \begin{array}{c} $	$\frac{3}{2}$ 5+1 = 1	$\begin{pmatrix} 1 - \frac{9}{2} & \frac{3}{2} \\ 1 - \frac{9}{2} & \frac{3}{2} \\ 1 - 18 & 6 \end{pmatrix}$						•
$\Rightarrow \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \text{ is an eigenvector for A with eigenvalue -1.}$ <u>that</u> : "Algorithm" to diagonalize a matrix: 1. Solve the polynomial equation det $(A - \lambda In) = 0 \implies Get$ eigenvalues $\lambda_1, \dots, \lambda_K$. 2. Find nonzero vectors in $Ker(A - \lambda_2 In), \dots, Ker(A - \lambda_K In)$ forming a basis $\mathcal{B} = v_{as}, \dots, v_{n}$. (* Warning Step 2 may fail!)	· ·	• • •		• • •		• • •	-→ÎL - Ī -→ĨL - Ī	$\begin{array}{c} 1 & -\frac{1}{3} \\ 0 & 0 \\ \end{array}$	· · · · ·	· · · ·	•
that: "Algorithm" to diagonalize a matrix: 1. Solve the polynomial equation det $(A - \lambda In) = 0$ ~ Get eigenvalues $\lambda_1, \dots, \lambda_K$. 2. Find nonzero vectors in $Ker(A - \lambda_2 In), \dots, Ker(A - \lambda_K In)$ forming a basis $B = v_{a_3}, \dots, v_n$ (! Warning Step 2 may fail!)		Ker(A	-XIn) =		ter y =	$= \operatorname{Span}_{1} \left(\begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right)$) ~ Bo	$Sis: \begin{pmatrix} 1\\ 3\\ 1 \end{pmatrix}$			•
 Solve the polynomial equation det (A-λIn) = 0 ~ Get eigenvalues λ₁, , λ_K. Find nonzero vectors in Ker(A-λ₂In),, Ker(A-λ_KIn) forming a basis B = v₁,, v_n. (! Warning Step 2 may fail!)) is an	ergenvector	for A	with eigenva	Jue -1.	· · ·			•
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2. Find nonzero vectors in Ker(A-l_In),, Ker(A-l_In) forming a basis B = v_1,, vn "an eigenbasis" (! Warning Step 2 may fail!)											•
· · · · · · · · · · · · · · · · · · ·	́⊥∢	Jolve The	polynomial	equation		−−λ=0)=0		eigenvalue	h the second dec.		•
· · · · · · · · · · · · · · · · · · ·	×	tind nonzeu	o vectors	in Ner($(A - \lambda_2 \perp_n)$, [\er (A	$-\lambda_{\kappa} \perp_{n}$	forming a	basis do = "an eign	V25>N enbasis	In -
3. Let $S = S_{B \rightarrow e} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$. Then $D = S A S$ is diagonal with entries λ_{2, \cdots, N_K} .	• •	·	Siep X	may Jan	()	• • •		• • •	• • • •	• • •	•
(some eigenvalues may be repeate	3.	Let S = S	S→e ⁼	(v,v.). (1 1).	Then I) = S ⁻ AS	is diago				

Exploring where our "Algorithm" can fail
Definition 3: Let $\lambda \in \mathbb{R}$, $T: \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation with matrix A. Then the λ -eigenspace of T
is $E_{\lambda} = Ker(A - \lambda I_n)$. Equivalently, E_{λ} is the subspace of all the vectors $v \in \mathbb{R}^n$ such that $Av = \lambda v$.
Example 2: Consider the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This represents a 90° rotation:
Intuitively, this has no eigenvectors. Let's perform the "algorithm".
1. Char puly: det $\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$ = $\lambda^{e} + 1$. This has no real roots. Thus A has no eigenvalues
-> There are no eigenvectors. => $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable.
Example 3 Consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This represents a shear :
Intuitively, the only eigenvectors lie on the x-axis. Let's perform the "algorithm".
1. Char poly: det $\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$. The only rost is $\lambda = 1$.
2. Let us find bases for the E's. Our only E_{λ} is $E_{4} = \text{ter}(A-4 \cdot I_{n}) = \text{ter}\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$.
Now Ker $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{cases} y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$
$= \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2} \qquad \begin{pmatrix} y \\ o \end{pmatrix} = \begin{pmatrix} o \\ o \end{pmatrix} $
$= \int \left(\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2} \\ \Rightarrow y = 0 \right)$
$= \operatorname{Span}\left(\binom{4}{0}\right).$
Unfortunately, this is a 1-dimensional subspace of \mathbb{R}^2 , hence any basis for it will only have
1 vector. Since E_1 is the only eigenspace, this means we will not be able to find an eigenbasis
\Rightarrow $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.
Remark: Examples 2 and 3 are the two kinds of things that can go wrong.
Example 4: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (har poly det $\begin{pmatrix} 1 - \lambda & 0 & 1 \\ 0 & 2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix}$ = $(1 - \lambda)^2 (2 - \lambda)$ => Eigenvalues: $\lambda = 1$, $\lambda = 2$.
2) $E_1 = \ker \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} \binom{\pi}{3} \\ \binom{\pi}{3} \end{pmatrix} \longrightarrow \dim = 1$
$E_2 = \operatorname{Ker} \begin{pmatrix} -1 & \circ & 1 \\ \circ & \circ & \circ \\ \circ & \circ & \circ \\ 0 & 0 & -1 \end{pmatrix} = \operatorname{Span} \left(\begin{pmatrix} 0 \\ 1 \\ \circ \end{pmatrix} \right) \rightarrow \dim = 1 \qquad \text{eigenbasis}$
$\Rightarrow \begin{pmatrix} a \lor a \\ \partial z & o \\ \lor & 0 \end{pmatrix} \text{ is mat diagonalizable}$

The following theorem should not be surprising at this point:
Theorem 1: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then the following stakements
ave equivalent
1) A is diagonalizable
2) There exists a basis of R" consisting of eigenvectors for A
3) The dimensions of the eigenspaces add up to n.
You may worry that the bases for E_{λ} and $E_{\lambda'}$ are linearly dependent
The following theorem says this connot happen.
Theorem 1: If A is an nxn matrix, v, w $\in \mathbb{R}^n$ and $c_s, c_z \in \mathbb{R}$ with $c_1 \neq c_z$ satisfy $Av = c_zv$ and $Aw = c_zv$
then v and w are linearly independent.
Proof: Suppose $\lambda v + \mu w = 0$ (*) Then $A(\lambda v + \mu w) = A \cdot 0 \Rightarrow \lambda c_1 v + \mu c_2 w = 0$ Subtracting these get Multiplying (*) by c_2 , we get $c_1 \lambda v + c_1 \mu w = 0$ $\mu(c_1 - c_2) w = 0$
Nultichen (x) hur no cot a hur ann = 0
Multiplying (*) by c_1 , we get $c_1 \lambda v + c_1 \mu w = 0$ $\mu(c_1 - c_2)w = 0$
Since $c_1 - c_2 \neq 0$, $\mu w = 0 \implies \mu = 0$. Similarly, $\lambda = 0$. Thus v and w are $1, i = 0$.
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Since $c_1 - c_2 \neq 0$, $\mu w = 0 \implies \mu = 0$. Similarly, $\lambda = 0$. Thus v and w are $1, i = 0$. In-class exercises:
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Since $C_i - C_2 \neq 0$, $\mu w = 0 \Rightarrow \mu = 0$. Similarly, $\lambda = 0$. Thus v and w are $1.i \cdot a$. In-class exercises: 1. Determine whether the following matrices are diagonalizable and diagonalize them if possible: 1) $\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$ 2) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$