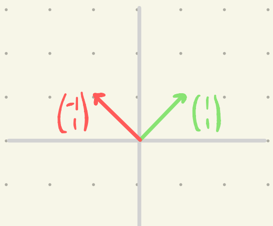
Lecture 11

Recall: a basis for $\mathbb{R}^n$ is a set of n vectors which are linearly independent and span $\mathbb{R}^n$.

Discussion: Theorem 2, Lecture 8: If $v_1, \ldots, v_n$ form a basis of $\mathbb{R}^n$, and $v \in \mathbb{R}^n$, then there exists a unique set of scalars $\lambda_1, \ldots, \lambda_n$ only depending on v such that $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$.

This means that we can use <u>any</u> basis to give our "coordinates in":

Example 1: The vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ form a basis (since they are l.i.)



Take a vector $v = \begin{pmatrix}3\\-2\end{pmatrix}$. Then $v = \lambda\begin{pmatrix}1\\1\end{pmatrix} + \mu\begin{pmatrix}-1\\1\end{pmatrix}$ for some unique (λ, μ).

This is a system of equations: $\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \cdot \begin{pmatrix}\lambda\\\mu\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$

Since $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$ form a basis, this has a unique solution: $\begin{pmatrix}\lambda\\\mu\end{pmatrix} = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}^{-1} \cdot \begin{pmatrix}3\\-2\end{pmatrix}$

The inverse of $\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$ turns out to be $\begin{pmatrix}1/2 & 1/2\\-1/2 & 1/2\end{pmatrix}$. So in this case $\begin{pmatrix}\lambda\\\mu\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\-1/2 & 1/2\end{pmatrix} \cdot \begin{pmatrix}3\\-2\end{pmatrix}$

$$= \begin{pmatrix}1/2\\-5/2\end{pmatrix}.$$

Let's check this:

$$\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{-5}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$$

So "in the basis" $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$, the vector $\begin{pmatrix}3\\-2\end{pmatrix}$ has "coordinates" $\begin{pmatrix}1/2\\-5/2\end{pmatrix}$.

<u>Definition 1</u>: Let $\mathcal{B}$ be an (ordered) basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$. Write $v = c_1 v_1 + \ldots + c_n v_n$.

Then the $\mathcal{B}$-coordinate vector of v is $[v]_{\mathcal{B}} = \begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$.

Theorem 1: Let $\mathcal{B}$ be a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$. Then $[v]_{\mathcal{B}} = \begin{pmatrix}| & & |\\v_1 & \cdots & v_n\\| & & |\end{pmatrix}^{-1} \cdot v$.

Definition 2: We will write $\mathcal{E}$ for the canonical basis $e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, e_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$.

Remark: Clearly, if $v = \begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}$ then $[v]_{\mathcal{E}} = \begin{pmatrix}a_1\\\vdots\\a_n\end{pmatrix}$.

Definition 3: The matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{-1}$ changes coordinates from $\mathcal{E}$ to $\mathcal{B}$, and so we will denote it $S_{\mathcal{E}\to\mathcal{B}}$

Remark: why is it $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{-1}$ and not $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$?

The matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ sends $\underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=[v_1]_{\mathcal{B}}}$ to $\underbrace{v_1}_{=[v_1]_{\mathcal{E}}}$, so it goes the other way.

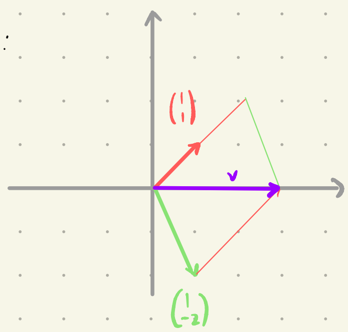
Definition 4: The matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ changes coordinates from $\mathcal{B}$ to $\mathcal{E}$, and so we will denote it $S_{\mathcal{B}\to\mathcal{E}}$

Example 2: Let $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a basis $\mathcal{B}$.

- Suppose $[v]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What is $[v]_{\mathcal{E}}$?

  $[v]_{\mathcal{E}} = S_{\mathcal{B}\to\mathcal{E}}[v]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}[v]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [v]_{\mathcal{E}}$.

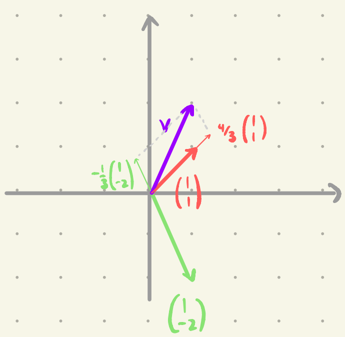
  Picture:



- Suppose $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \left(= [v]_{\mathcal{E}} \right)$. What is $[v]_{\mathcal{B}}$?

  $[v]_{\mathcal{B}} = S_{\mathcal{E}\to\mathcal{B}}[v]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}^{-1}[v]_{\mathcal{E}} = \begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 4/3 \end{pmatrix}$.

  Picture:



Observation: $S_{\mathcal{B}\to\mathcal{E}}$ and $S_{\mathcal{E}\to\mathcal{B}}$ are inverse to each other.

Discussion: Suppose we'd like to change bases from a basis B_1 to a basis B_2.

Then all we have to do is: $[v]_{B_1} \longmapsto S_{B_1\to e}[v]_{B_1} \mapsto S_{e\to B_2} S_{B_1 \to e}[v]_{B_1}$.

Definition 5: The change of basis matrix from a basis B_1 to another basis B_2 is $S_{e\to B_2} \cdot S_{B_1\to e}$.

It is denoted $S_{B_1 \to B_2}$.

Example 3: Let $B_1 = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ and $B_2 = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$. Then:

$$S_{B_1\to e} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad S_{e\to B_2} = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}^{-1} = \begin{pmatrix}-1 & 0\\ 0 & 1/2\end{pmatrix},$$

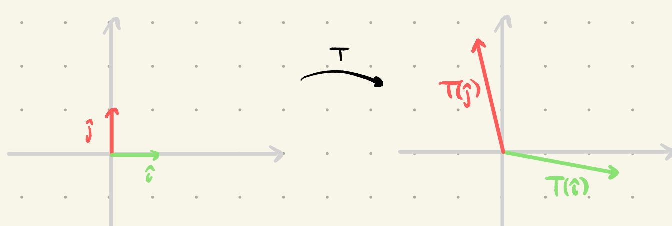
$$S_{B_1\to B_2} = \begin{pmatrix}-1 & 0\\ 0 & 1/2\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & -1\\ 0 & 1/2\end{pmatrix}.$$

Discussion: this explains how to change bases of individual vectors. The next step is how to change bases of <u>linear transformations</u>.

Important question you should be asking yourself: why change basis in the first place?

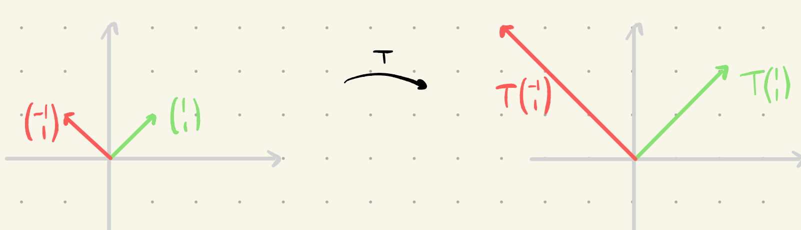
Answer: some bases are better than others!

Example 4: Consider the linear transformation with matrix $A = \begin{pmatrix}5/2 & -1/2\\ -1/2 & 5/2\end{pmatrix}$. This is hard to wrap my head around:



However consider what it does to the basis $B = \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$:

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$, $T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\-3\end{pmatrix}$ It scales them! In this alternative basis:



Much easier to think about!
+
Easier to compute with!

Crucial observation of today's lecture:

$$A = \underbrace{S_{\mathcal{B}\to e}}_{\substack{\text{put the coords}\\ \text{back to normal ones}}} \cdot \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\substack{\text{apply } T \text{ in that basis}\\ \text{(easier!)}}} \cdot \underbrace{S_{e\to\mathcal{B}}}_{\text{change coords to } \mathcal{B}\text{-coords}}$$

In turn, the (easier) matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ can be obtained as

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = S_{\mathcal{B}\to e}^{-1} \cdot A \cdot S_{e\to\mathcal{B}}^{-1} = S_{e\to\mathcal{B}} \cdot A \cdot S_{\mathcal{B}\to e}$$

This motivates:

Definition 5: Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A.

Then the matrix of T with respect to $\mathcal{B}$ (or $\mathcal{B}$-matrix of T) is $S_{e\to\mathcal{B}} \cdot A \cdot S_{\mathcal{B}\to e}$

Remark: writing $S = \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{\text{basis } \mathcal{B}}$, the matrix of T wrt $\mathcal{B}$ is $S^{-1}AS$.

More motivation:

Example 5: You're in a real life situation where you need to apply the same T many times, say 100 times. In other words you have a matrix A, for instance $A = \begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$, and you need to figure out A^{100}.

You notice it's taking forever, so you decide to be smart about it, and notice that

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{S} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}}_{S^{-1}}.$$

You realize: $(SAS^{-1})^{100} = S A \cancel{S^{-1} S} A \cancel{S^{-1}} \cdots \cancel{S^{-1} S} A S^{-1}$

$$= S A^{100} S^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

and you're done after 3 matrix multiplications!

Discussion: depending on the basis you choose, a linear transformation can be represented by very different looking matrices. However, they will all be related by change of basis matrices.

Definition 6: Let A and B be square $n \times n$ matrices. Then B is <u>similar</u> to A iff there exists an invertible matrix S such that $B = S^{-1}AS$. We denote it $B \sim A$.

Equivalently, B is similar to A iff B represents the same linear transformation as A, with respect to a different basis.

Theorem 2: The following hold:

1) Any square matrix A is similar to itself. $(A \sim A)$
2) If A is similar to B, then B is similar to A. $(A \sim B \Rightarrow B \sim A)$
3) If A is similar to B, and B is similar to C, then A is similar to C. $(A \sim B, B \sim C \Rightarrow A \sim C)$

<u>Proof</u>: 1) Taking $S = I_n$, $A = S^{-1}AS = I_n^{-1} A I_n = A$.

2) If A is similar to B, there exists an invertible matrix S_0 such that $A = S_0^{-1}BS_0$.
But then $B = S_0 A S_0^{-1}$, so it suffices to take $S = S_0^{-1}$: $B = S^{-1}AS$.

3) In-class exercise.

Recall that a matrix is diagonal iff it is of the form $\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$.

Definition 7: A matrix is <u>diagonalizable</u> iff it is similar to a diagonal matrix.

Questions for next time: when/how can we diagonalize a matrix?

In-class exercises:

1. Let $A = \begin{pmatrix} -\frac{11}{2} & \frac{3}{2} \\ -18 & 5 \end{pmatrix}$. You're given that $A \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $A \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

   Compute A^{100}, rounding to 3 decimal places.

2. Prove that $A \sim B$ and $B \sim C$ imply $A \sim C$.

1. Let $A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}$ Notice that $A\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Compute A^{100}, rounding to 3 decimal places.