Lecture 10

Plan for the next few lectures: deeper study of linear transformations

- Change of basis
- Eigenvalues, eigenvectors, diagonalization

Today: determinants

Discussion / opinion: the determinant (as we will see shortly) is a number associated to a linear transformation.

it is <u>not</u> a very useful computational tool, but it has an important theoretical value.

Usually, it is introduced earlier. In fact, determinants predate linear algebra!

We introduce them today because:

- They become necessary in what follows
- They are relatively light in complexity

So what is the determinant?

- 1×1 matrices

The determinant of a 1×1 matrix (a) is a. Boring.

- 2×2 matrices

The determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad-bc$

Let's see some examples:

Example 1: • $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(A) = 1\cdot1 - 0\cdot0 = 1$

• $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(A) = 3\cdot1 - 0\cdot0 = 3$

• $A = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} \Rightarrow \det(A) = 5\cdot7 - 0\cdot0 = 35$

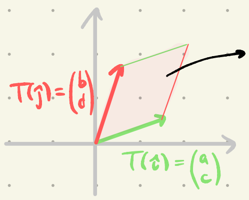
- $A = \begin{pmatrix} 5 & 9 \\ 0 & 7 \end{pmatrix} \Rightarrow \det(A) = 5 \cdot 7 - 9 \cdot 0 = 35$

- $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \Rightarrow \det(A) = ad$.

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(A) = -1$

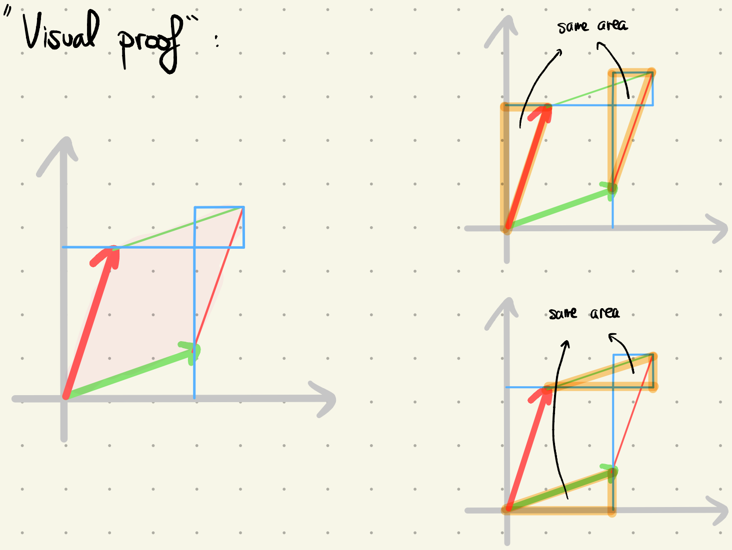
The idea: let's picture the matrix A as usual:

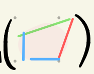
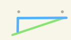
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



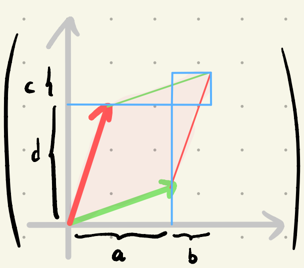
the "signed" area of this parallelogram is $\det(A)$

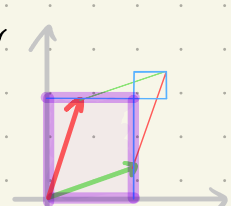
"Visual proof":



However this double counts Area() and includes the areas of  and .

Therefore

Area 

= Area 
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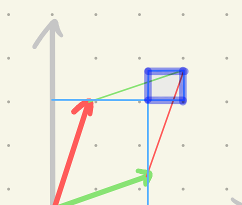
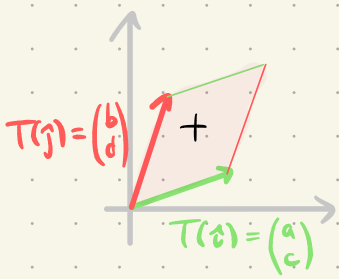
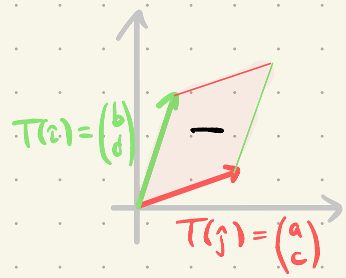
− Area 
bc

It is "signed" in the sense that





Reversing "orientation" changes the sign.

Important observation: A is invertible $\Leftrightarrow$ $T(\hat{i})$ and $T(\hat{j})$ are l.i.

$\Leftrightarrow$ $T(\hat{i})$ and $T(\hat{j})$ are not on the same line

2x2 matrices only!

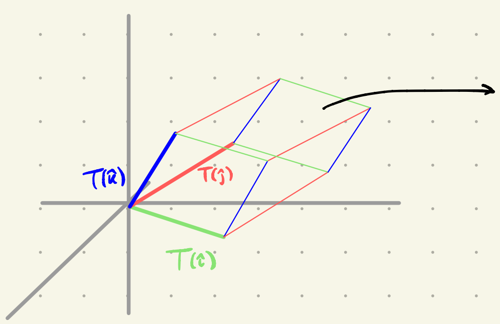
$\Leftrightarrow$ The signed area of the parallelogram is $\neq 0$.

$\Leftrightarrow$ $\det(A) \neq 0$.

This pattern holds in general, for square matrices.

In general, the determinant will be the "volume of the signed parallelepiped" formed by the vectors

$T\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right), \dots, T\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right)$:



Volume of this = $\det(A)$

Observe: $T(\hat{i}), T(\hat{j}), T(\hat{k})$ are l.i. $\Leftrightarrow$ they lie on the same plane

$\Leftrightarrow$ volume of the parallelepiped $= 0$

$\Leftrightarrow$ $\det(A) = 0$.

How to compute the determinant for a general square matrix?

Definition 1: Let A be a square matrix, and put it into rref. Let $s = \#\text{swaps}$ and $k_1, \dots, k_t$ the scalars that were used to divide the rows. Then, $\det(A) = \begin{cases} 0 & \text{if some row has no pivot} \\ (-1)^s k_1 \cdots k_t & \text{otherwise} \end{cases}$

Remark: Clearly A is invertible $\Leftrightarrow$ $\det(A) \neq 0$.

Remark: It suffices to do rref until putting A into triangular form: $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

(since no swaps or scalings will be performed after that point)

Example 2: • If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and $a \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{I} \to \frac{1}{a}\text{I}} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \xrightarrow{\text{II} \to \text{II} - c\cdot\text{I}} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix} \xrightarrow{\text{II} \to \frac{1}{d - \frac{bc}{a}} \cdot \text{II}} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \det(A) = (-1)^0 \cdot a \cdot \left(d - \frac{bc}{a}\right) = ad - bc$$

• If $a = 0$ then

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \xrightarrow{\frac{1}{c}\cdot\text{II}} \begin{pmatrix} 0 & b \\ 1 & \frac{d}{c} \end{pmatrix} \xrightarrow{\text{I} \leftrightarrow \text{II}} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & b \end{pmatrix} \xrightarrow{\text{II} \to \frac{1}{b}\text{II}} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$$

$$\det(A) = (-1)^1 \cdot c \cdot b = -bc = ad - bc$$

Example 3: $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{III} \to \text{III} + \text{I}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{\text{III} \to \text{III} - 2\cdot\text{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{pmatrix} \xrightarrow{\text{III} \to \frac{1}{-6}\text{III}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

So $\det(A) = (-1)^0 \cdot (-6) = 6$

Properties of the determinant.

Theorem 1: Let A and B be square $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.

Proof:

• If A is not invertible, $\text{Im}(A) \neq \mathbb{R}^n$, hence $\text{Im}(AB) \neq \mathbb{R}^n$, so AB is not invertible. Thus

$$\det(AB) = 0 = \det(A) \cdot \det(B)$$

• If B is not invertible, $\text{Ker}(B) \neq \{0\}$, hence $\text{Ker}(AB) \neq \{0\}$, so AB is not invertible. Thus

$$\det(AB) = 0 = \det(A) \cdot \det(B)$$

- If A, B are both invertible:

Recall that $\left(A \mid I_n\right) \xrightarrow{\text{Gauss}} \left(I_n \mid A^{-1}\right)$

Similarly, $\left(A \mid C\right) \longrightarrow \left(I_n \mid A^{-1}C\right)$ (Extra exercise)

In particular, if $C = AB$,

$$\left(A \mid AB\right) \xrightarrow{\text{Gauss}} \left(I_n \mid B\right) \rightsquigarrow s, k_1, \dots, k_t \text{ for } A$$

Now keep doing Gaussian elimination on $B \rightsquigarrow s', k_1', \dots, k_{t'}'$ for B:

$$\left(* \mid I_n\right)$$

Looking at the RHS of the divider, we have put AB into rref, hence

$$\det(AB) = (-1)^{s+s'} k_1 \cdots k_t \cdot k_1' \cdots k_{t'}' = (-1)^s k_1 \cdots k_t \cdot (-1)^{s'} k_1' \cdots k_{t'}' = \det(A) \cdot \det(B). \quad \square$$

Corollary: For any square matrix A,

- $\det(A^n) = \det(A)^n$
- $\det(A^{-1}) = \det(A)^{-1}$

Proof: The first one is clear. For the second one: $I_n = AA^{-1} \Rightarrow 1 = \det(A) \cdot \det(A^{-1})$.

Remark: Swapping two rows of A changes the determinant to $-\det(A)$.

Alternative approach to compute the determinant.

Theorem 2 (Laplace expansion) Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ matrix.

Then $\det(A)$ can be computed recursively as:

$$a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{pmatrix} + \dots + (-1)^{n+1} \det \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{pmatrix}$$

Proof: omitted.

Example 4: let's find the determinant of $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$.

$$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} = 1 \cdot (-5) - 0 \cdot 2 - 1 \cdot 1 = -6$$

In fact this can be done along any row or column:

Theorem 2:

Laplace expansion (or cofactor expansion)

We can compute the determinant of an $n \times n$ matrix A by Laplace expansion down any column or along any row.

Expansion down the jth column:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Expansion along the ith row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Definition 2: Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ be an $m \times n$ matrix. Then the transpose of A is

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

Theorem 3: Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be a square matrix. Then $\det(A) = \det(A^T)$.

Proof: • Case $n=2$: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ ✓

• Case $n=3$:

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \underset{\text{Laplace, 1st col}}{=} a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21} \cdot \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31} \cdot \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

= by n=2   = by n=2   = by n=2

$$\det(A^T) = \det\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \underset{\text{Laplace, 1st row}}{=} a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} - a_{21} \cdot \det\begin{pmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{pmatrix} + a_{31} \cdot \det\begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix}$$

The general case is similar. □

In-class exercises:

1. Compute the determinant of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

   - Using rref
   - Using the Laplace expansion

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. Compute $\det((A^{-3})^T)$.

3. Is it true that $\det(A+B) = \det(A) + \det(B)$?