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ective 1				• • •	• •		•	• •
· · · · · · · · · · · ·							•	
systems of linear	equations							
		· · · · · ·						
Example 1 let's s	she the following	system of equations	÷					
	ι Ο							
x + 3y = 5	. Π-2.T . X+	$+3\gamma = 5$					٠	
		· · · · · ·		• • •				
$x^{2x} - y^{2x} = 3$		-7 y = -+ \			• •	• • •	٠	
				• • •				
	· II ->	x + 3y = 5		• • •				
		x = 1 = 14						
					• •	• • •	٠	
					• •	• • •	٠	
	T→ +-3·1	\sim -2		• • •	• •		•	• •
	`	\wedge μ μ μ μ	 D	· · · ·			•	• •
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Inthy finear?					• •	• • •	٠	
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· · · · · · ·	· · · · · · · ·	· · · · · · ·		· · ·	· ·		•	••••
	is the equation for a	line		· · · ·	· ·	· · ·	•	· ·
x + 3y = .5	is the equation for a	line		(5,0)	· · ·	· · · ·		· · ·
	is the equation for a	line		(5,0)	· · · · · · · · · · · · · · · · · · ·	· · · ·	•	· · ·
x + 3y = .5	is the equation for a t	lire		(5,0) 	· · · · · · · · · · · · · · · · · · ·	· · · ·		· · ·
x .+ 3y .= .5	is the equation for a t	line	(0,5) 	(5,0) , , , , , , , , , , , , , , , , , , ,	· · · · · · · · · · · · · · · · · · ·	· · · ·	•	· · · · · · · · · · · · · · · · · · ·
	is the equation for a t	$\lim_{x \to 0} x = 0 \Rightarrow$	$(0, \frac{5}{5})$	(5,0) 	· · · · · · · · · · · · · · · · · · ·	· · · · ·	•	· · · · · · · · · · · · · · · · · · ·
x + 3y = .5	is the equation for a l	line $x=0$	$(0, \frac{5}{3})$	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · ·		· · · · · · · · · · · · · · · · · · ·
	is the equation for a t	$\lim_{x \to 0} \frac{1}{x} = 0 = 0$	$(0, \frac{5}{5})$	(5,0) (5,0)	 . .<	· · · · ·		· · · · · · · · · · · · · · · · · · ·
	is the equation for a t	$\lim_{x \to 0} \frac{1}{y} = 0$	$(0, \frac{5}{5})$	(5,0) 	 . .<	· · · · · · · · · · · · · · · · · · ·		· · · · · · · · · · · · · · · · · · ·
x + 3y = .5	is the equation for a 1	$\lim_{x \to 0} \frac{1}{x} = 0 = 0$	$(0, \frac{5}{3})$ $(\gamma = \frac{5}{3}$ $(\gamma = \frac{5}{3})$	(5,0) (5,0) 	 . .<	· · · · · · · · · · · · · · · · · · ·		· · · · · · · · · · · · · · · · · · ·
$x + 3y = 5$ $Same \int 0f = 2x - y$	is the equation for a 1	$\lim_{x \to 0} \frac{1}{x} = 0 = 0$	$(0, \frac{5}{3})$	(5,0) (5,0) 	 . .<	· · · · · · · · · · · · · · · · · · ·		· · · · · · · · · · · · · · · · · · ·
x + 3y = 5 Same for $2x - y$	is the equation for a $y = 3$:	line $x=0$	$(0, \frac{5}{2})$	(5,0) 	 . .<	· · · · · · · · · · · · · · · · · · ·		· · · · · · · · · · · · · · · · · · ·
x + 3y = 5 Same for $2x - y$	y = 3:	fire $x = 0 \Rightarrow$ $x = 0 \Rightarrow$ $y = 0 \Rightarrow$ $(\frac{3}{2}, 0) \times 1$	$(0, \frac{5}{3})$ $(\gamma = \frac{5}{3}$ $(\gamma = \frac{5}{3})$ $(\gamma = \frac{5}{3})$	(5,0) (5,0)	 . .<	 . .<		
x + 3y = 5	is the equation for a $y = 3$:	$\begin{cases} x = 0 \\ x = 0 \\ y $	$(0, \frac{5}{5})$	(5,0) (5,0)	 . .<	 . .<		
x + 3y = .5 Same for $2x - y$	is the equation for a $y = 3$:	line $x = 0 = 3$ y = 0 = 3 (0, -3)	$(0, \frac{5}{3})$ $(1, \frac{5}{3}, \frac{5}{3})$ $(2, \frac{5}{3}, \frac{5}{3})$ $(3, \frac{5}{3}, \frac{5}{3})$ $(3, \frac{5}{3}, \frac{5}{3})$					
x + 3y = 5 Same for $2x - y$	y = 3:	$\lim_{x \to 0} \frac{1}{x} = 0 = 0$	$(0, \frac{5}{3})$ $(\gamma, = \frac{5}{3})$ $(\gamma, = \frac{5}{3})$ $(\gamma, = \frac{5}{3})$	(5,0) (5,0) 	 . .<			
x + 3y = 5 Same for $2x - y$	y = 3:	$\begin{cases} x = 0 \\ y $	$(0, \frac{5}{3})$	(5,0) (5,0)	 . .<			

Then the solution of t	te system is proc	usly the	intersection	of the	two f	nos :	· ·	• •	•	•	•
· · · · · · · · · · · · · · · · · · ·						•			•	•	•
					• •	•	• •	• •	•	•	•
· · · · · · · · ·	(2,1)	· · · · ·	· · · ·		• •	•	• •		•	•	•
	(<u>3</u> ,0) (5,0)					•			•	•	•
	0,-3) · · · ·								٠		
· · · · · · · · · · /			· · · ·			•			•	•	•
 .	• • • • •					•			•	•	•
. This "algorithm" a	ilso works for	systems with	h more va	anables	and	more	equat	tians :	٠		
Example 2		· · · ·	· · · ·		• •	•			•	•	•
x + 2y + 3z =	39	x + 2y + 3z	- = 39 (•			•	•	•
3 \pm 2 $=$	$\begin{array}{c c} \underline{\mathbb{I}} \rightarrow \underline{\mathbb{I}} - \underline{\mathbb{I}} \\ \underline{34} & \longrightarrow \end{array}$		5		• •		• •				•
		y - c				•	· ·		•	•	•
3x + 2y + 2 =		3x + 2y + i	2 = 20			•		• •	•	•	•
		v v 7 v × 3 a	 								•
	Ⅲ→Ⅲ-3 Ⅰ	×+ 29 + 0 e		• •		•			•	•	•
		· · · · · · · ·	=-2			•			•	•	•
		· · -44 - 8	$r_{2} = -9$						٠		•
			· · · · ·		• •	•			•	•	•
	Ⅲ->Ⅲ+4-∏	X+ 2y + 37	+ = 39			•			•	•	
		· · · · · · · · · · · · · · · · · · ·	, ≃ ⁻ - 5 , ,								
		- 12	2 = -		• •	•	• •		•	•	•
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		v . c -	<u> </u>						*		
			. –								
· · · · · · · · ·	Ĩ →Ĩ-2 Ī 	· · · · + · · ·	=-2			•			•	•	•

			X + 5 7 = 49				
• •			$y_{1} - z_{2} = z_{1} - S_{1}$			• • •	
• •			2 - 625				
• •					• • •		• • •
• •				• • • •	• • •		• • •
			X + 57 = 49				
		Ĩ→Ĩ+ĨĹ				• • •	
			$y_{1} = y_{1} = y_{1} = y_{1} = y_{2} = y_{1} = y_{1} = y_{2} = y_{1} = y_{2} = y_{1} = y_{2} = y_{1} = y_{2} = y_{2} = y_{1} = y_{2} = y_{2} = y_{1} = y_{1} = y_{2} = y_{1} = y_{1$		• •		
			2 = 925				
• •	• • •			• • • •	• • •	• • •	• • •
			X = 2.75				
		I→I-5Ⅲ				(2 75 4	25.9.25)
• •			$\gamma = 4.23$		(XIT, FI-	e solution	to the sustem
			2 = 925]];	s the only	بر ایدر اید	i is incogain
• •	• • • •				• • •		• • •
Pernark 1:	geometrical	y, an equation	on like $x + 2y + 27 = 2$	is the equa	tion of a p	plane in	R
<u>Pernark 1</u> : Usually	geometricall	y, an equations intersect in a	on like $x + 2y + 2z = 2$ line, and three planes in	is the equation is the equation in a provident terrect in a provident terrect in a provident terrect in a provident terrect te	tion of a p point (see	below)	R ^s .
<u>Pernark 1</u> Usually",	geometricall two planes	y, an equations intersect in a	on like $x + 2y + 2z = 2$ line, and three planes in	is the equation $a_{\rm F}$	tion of a p point (see	plane in below)	R ^s .
<u>Pernark 1</u> Voually [°] , In examp	geometricall two planes le 2, the	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa terrect in a p h interrect	tion of a f point (see in the	plane in below) point ((2.75,4 25,9 2
<u>Pernark 1</u> Usually [°] , In examp	geometricall two planes le 2, the	Y, an equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa terrect in a p h interrect	tion of a f point (see in the	plane in below) point ((2.75,4 23,9 2
<u>Zemark 1</u> Usually", In examp	geometricall two planes le 2, the	Y, an equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa terrect in a p h interrect	tion of a f point (see in the	plane in below) point ((2.75,4.25,9.2
<u>Pernark 1</u> Voually", In examp	geometricall two planes le 2, the	Y, an equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes whic	is the equa terrect in a p h interrect	tion of a f point (see in the	plane in below) point ((2.75,423,92
<u>Zemark 1</u> Usually", In examp	geometricall two planes le 2, the	Y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa tersect in a p h intersect	tion of a p coint (see	plane in below) point ((2.75,423,92
<u>Zemark 1</u> Usually", In examp	geometricall two planes le 2, the	Y, an equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa tersect in a p h intersect	tion of a p coint (see	plane in below)	(2.75, 4.25, 9.2
<u>Remark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equations the equation of the equation	tion of a p coint (see	plane in below)	R ⁵ (2.75, 4.25, 9.2
<u>Remark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equation in tersect in a provident of the sector of	tion of a p coint (see	plane in below)	R ⁵ (2.75, 4. 23, 9.2
<u>Pernark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equation in tersect in a provide the intersect of the solution numbers	tion of a p coint (see	plane in below)	R ^s (2.75, 4 23, 9 2
<u>Pernark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equation in tersect in a particular solution numbers of the solution nu	tion of a point (see	plane in point (R ^s (2.75, 4 23, 9 2
<u>Pernark 1</u> Usually", In examp	geometrical	y, an equations three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa terrect in a p h interrect	tion of a point (see	plane in point (R ^s (2.75, 4.25, 9.2
<u>Pernark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which	is the equa terrect in a p h interrect Solution ntersection	tion of a p coint (see	plane in point (R ^s (2.75, 4.25, 9.2
<u>Pernark 1</u> Usually", In examp	geometricall two planes le 2, the	y, an equations three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which is a set of the planes in the planes is the pl	is the equation terrect in a p h interrect	tion of a p coint (see	plane in point (R ^s . (2.75, 4.23, 9.2
<u>Pernark 1</u> Usually", In examp	geometrical	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which is a set of the planes which is a set of the planes is	is the equation terrect in a p h intersect	tion of a p coint (see	plane in point (R ^s . (2.75, 4.23, 9.2
<u>Pernark 1</u> Usually", In examp	geometrical	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which is a set of the planes in the planes is the planes in the planes is the planes is the planes is the plane	is the equation terrect in a p h intersect	tion of a p coint (see	plane in point (R ^s . (2.75, 4.23, 9.2
<u>Zemark 1</u> Usually", In exampl	geometrical	y, an equations intersect in a three equations	on like $x + 2y + 2z = 2$ line, and three planes in determine three planes which is a set of the planes in the planes is a set of the planes is a set of the plane is a set of	is the equation terrect in a p h intersect	tion of a p coint (see	plane in i	R ^s . (2.75, 4.23, 9.2

Permark 2 Sometimes the system doesn't have a unique solution, a	or even any solutions at all.
Example 3 (No solutions)	
$\frac{1}{2} = \frac{1}{2} \sqrt{\pi} \pi 2^{-1} = \frac{1}{2} \sqrt{2} = \frac{1}{2}$	
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	
Impossible! No solutions	
Geometrically	
· · · · · · · · · · · · · · · · · · ·	
The lines are parallel	
· · · · · · · · · · · · · · · · · · ·	
· · · · · · · · · · · · · · · · · · ·	
· · · · • · · · · · · · · · · · · · · ·	
Evaluare 4 (InProvident and a literary)	
Crambe 1 (infinited many solutions)	
x . + . y . + . z . = . 4	
$2x + 3z = 6 \qquad \qquad$	
$2y_1 - z_2 = 2$	
x + y + z = 1	
$ \square \rightarrow \square + \square $	
0 = 0	
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	
$\mathbb{I} \to \frac{1}{2}\mathbb{I}$	
0 = 0	
· · · · · · · · · · · · · · · · · · ·	
$X + 3_7 = 0$	Can't progress further. Any value
$-\frac{1}{2}z = 1$	for z will give a valid solution.
	• • • • • • • • • • •

Therefore, we say that 2 is a free variable, because it may take any real value. The variables x, y
are determined (in this case) by the value of 2. The set of solutions is
$\left(-\frac{3}{2}t, 1+\frac{1}{2}t, t\right)$ $\left[t \in \mathbb{R}\right]$ [Geometrically: three planes intersecting in a line]
Theorem 1: A system of linear equations either has
• 1 solution
 No solutions
· Infinitely many solutions
Remark 3. Netre that we have been performing operations on surfaces of equations as if the entries were
<u>Permin 3</u> Notice that we have been performing operations of systems of equations as if the entries were
on a table. Moreover, we don't really need to write x, y, z over and over.
Definition 1: An m-by-n matrix is a table of real numbers with m rows and n columns:
$m=3$ $\begin{pmatrix} 2 & -3 & 4 & 2.5 \\ 14 & 2 & -1 & 2 \end{pmatrix}$
(-3, 4, 0, 4)
Definition 2 An augmented matrix is a matrix with an extra column and a divider.
(2 -3 4 2.5 1).
To each assemented matrix corresponds a unique sustem of linear equations and vice-versa.
to care a finance in any concerned on the stand of the standard and the standard and the
$\begin{pmatrix} 2 & -3 & 4 & 2.5 & 1 \\ 1.1 & 0 & -1 & 2 & -2 \\ \end{pmatrix} \xrightarrow{2x - 3y + 4z + 2.5w} = 1$
$\begin{pmatrix} -3 & 1 & 0 & 1 & 3 \end{pmatrix}$ $-3x + 1y + 0 = 3$
Then, as before, we can solve the system by performing row operations to the augmented matrix:
· Add a multiple of a row to another row
the three "elementary row operations"
· Mutiply a row by a nonzero number
· Swap two rows

	,			1 4			
	$2x_{1} - \gamma = 3$	· · · · · · ·	(2-1				
n all of our examples	, we obtained	an augmented	d matrix of	the form			
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	0 * * 0 * + 0 * + 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0 * * 0	-reduced eched	bn form	(RREF)	. More for	rmally :	
Reduced row-echelon form					· · · ·	· · · · ·	
A matrix is said to be in <i>reduce</i> following conditions:	d row-echelon form (rref) i	f it satisfies all of the					
 a. If a row has nonzero entries <i>leading</i> 1 (or <i>pivot</i>) in this ro b. If a column contains a lead are 0. 	s, then the first nonzero en w. ing 1, then all the other en	try is a 1, called the					
c. If a row contains a leading further to the left	1, then each row above it	contains a leading 1	l				
Condition c implies that rows o	f 0's, if any, appear at the b	ottom of the matrix.					
veorem 2: Any matrix	can be put into	RREF by	elementary	row opera	ations.	· · · · ·	
<u>veorem 2</u> : Any matrix <u>elimition</u> : The process called Gaussian elimina	can be put into of turning a n	natrix into f	elementary REF by p	erforming rou	ations. U operations	· · · · · ·	
veorem 2: Any matrix efinition: The process called Gaussian elimina	can be put into of turning a n	natrix into f	elementary REF by p	erforming rou	ations. U operations	· · · · · ·	
reprem 2: Any matrix e/inition: The process called Gaussian elimina - class exercise session	can be put into of torning a n	ndrix into f	elementary	erforming you	ations. U operations		
rearem 2: Any matrix e <u>finition</u> : The process called Gaussian elimina - class exercise session 1 White down the ave	can be put into of torning a n non	natrix into f	elementary REF by p	erforming yrou	ations. u operations		
rearem 2: Any matrix efinition: The process called Gaussian elimina 1 - class exercise session 1. Write down the au x + y = 5	can be put into of torning a n fion	natrix into f	elementary REF by p te following s	ystem of linea	ations. u operations ir equations:		
represent 2: Any matrix effinition: The process called Gaussian elimina 1 - class exercise session 1. Write down the aver x + y = 5 x + z = 7 y + z = 8	can be put into of torning a n non	natrix into f	elementary REF by p the following s	erforming rou ystem of linea	ations. u operations ir equations:		
neorem 2: Any matrix efinition: The process called Gaussian elimina 1 Write down the au x + y = 5 x + z = 7 y + z = 8 2 Perform Gaussion elimina	can be put into of torning a n (ion junented matrix a	notrix into f	elementary REF by p the following s	erforming rou ystem of linea to solve th	ations. v operations ir equations: he system.	Check that	
<u>represen 2</u> : Any matrix <u>efinition</u> : The process called Gaussian elimina 1. Write down the aver x + y = 5 x + z = 7 y + z = 8 2. Perform Gaussian elimination x + y = 5 x + z = 7 y + z = 8	can be put into of torning a m tion immented matrix a minution on the matrix	notrix into f	elementary REF by p the following s in 4 in orde	erforming rou ystem of linea to solve th	ations. U operations ir equations: he system.	Check that	