Mock Linear Algebra Midterm

1. (20 points) For the following matrices, determine if the corresponding linear transformation is injective, whether it is surjective, and whether it has an inverse. Justify your answers. In the case that it has an inverse, compute it.

   (a) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. (20 points) Find the values of $\lambda, \mu \in \mathbb{R}$ such that the matrices $A = \begin{pmatrix} 1 & \lambda \\ \lambda - 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ commute. (Two matrices commute if $AB = BA$.)

3. Let S_1 and S_2 be subspaces of $\mathbb{R}^n$, and let

$$S_1 \cap S_2 = \{v \in \mathbb{R}^n : v \in S_1 \text{ and } v \in S_2\}$$

   Prove or disprove: $S_1 \cap S_2$ is a subspace of $\mathbb{R}^n$. Let

$$S_1 \cup S_2 = \{v \in \mathbb{R}^n : v \in S_1 \text{ or } v \in S_2\}$$

   Prove or disprove: $S_1 \cup S_2$ is a linear subspace of $\mathbb{R}^n$.

4. (20 points) Give an example of an injective linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$. Draw your linear transformation in the way we have done it in the course, identifying in your picture: where the basis vectors go, the kernel and the image of the transformation. Explain also how to obtain the image of the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in terms of your drawing.

5. (20 points) Determine whether the following statements are true or false. Justify your answers: give a proof if they are true and give a counterexample if they are false.

(a) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ may have rank 4.

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ of rank 3 must be injective.

(c) If AB is the identity matrix, then A and B must be square matrices.

(d) If S and T are subspaces of $\mathbb{R}^n$ and S is contained in T, then $\dim(S) \leq \dim(T)$.

(e) If $x_0 \in \mathbb{R}^n$ is a solution of the equation $Ax = b$ where A is an $m \times n$ matrix, and $x_1 \in \mathbb{R}^n$ is in the kernel of A, then $x_0 + x_1$ is another solution to the equation $Ax = b$.

6. (**Extra:** 10 points) A 3×3 matrix is diagonal if it is of the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Prove that if a diagonal matrix D commutes with every other 3×3 matrix, then D is of the form $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. In other words, D must be a multiple of the identity matrix.