Mock final Solutions

1\. (a) $A = \begin{pmatrix} 4 & 4 \\ -1 & 5 \end{pmatrix}$

A is not symmetric $\Rightarrow$ A is <u>not</u> orthogonally diagonalizable.

Eigenvalues: $\text{charpoly}(A) = \det\begin{pmatrix} 4-\lambda & 4 \\ -1 & 5-\lambda \end{pmatrix} = (4-\lambda)(5-\lambda) + 4 = \lambda^2 - 9\lambda + 24$

$$\Rightarrow \lambda = \frac{81 \pm \sqrt{81 - 4\cdot 24}}{2}$$

Since $81 - 4\cdot 24 < 0$, A has no real eigenvalues $\Rightarrow$ A is <u>not</u> diagonalizable.

(b) $A = \begin{pmatrix} 0 & 2 & & \\ -2 & 0 & & \\ & & 0 & 3 \\ & & -3 & 0 \end{pmatrix}$

A is not symmetric $\Rightarrow$ A is <u>not</u> orthogonally diagonalizable.

Eigenvalues:
$$\text{charpoly}(A) = \det(A - \lambda I_4) = \det\begin{pmatrix} -\lambda & 2 & & \\ -2 & -\lambda & & \\ & & -\lambda & 3 \\ & & -3 & -\lambda \end{pmatrix}$$

$$= (\lambda^2+4)(\lambda^2+9)$$

↳ This has no real roots $\Rightarrow$ A is <u>not</u> diagonalizable.

(c) $A = \begin{pmatrix} -2 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$

A is not symmetric $\Rightarrow$ A is <u>not</u> orthogonally diagonalizable.

Eigenvalues: $\text{charpoly}(A) = \det\begin{pmatrix} -2-\lambda & -1 & 1 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -3-\lambda \end{pmatrix}$

$$= (-2-\lambda)\cdot\det\begin{pmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{pmatrix} + \det\begin{pmatrix} 2 & -2 \\ 2 & -3-\lambda \end{pmatrix} + \det\begin{pmatrix} 2 & 1-\lambda \\ 2 & 2 \end{pmatrix}$$

$$= (-2-\lambda)\cdot\Big((1-\lambda)(-3-\lambda)+4\Big) + \Big(2\cdot(-3-\lambda)-2\cdot(-2)\Big) + \Big(4-2\cdot(1-\lambda)\Big)$$

$$= (-2-\lambda)\cdot\big(\underbrace{\lambda^2+2\lambda-3+4}_{\lambda^2+2\lambda+1}\big) + (-6-2\lambda+4) + (4-2+2\lambda)$$

$$= -\lambda^3-2\lambda^2-\lambda-2\lambda^2-4\lambda-2-2\lambda-2+2+2\lambda$$

$$= -\lambda^3-4\lambda^2-5\lambda-2 \quad (*)$$

Observe: $\lambda=-2$ is a solution: $+2^3-4\cdot 2^2+5\cdot 2-2 = 8-16+10-2=0$.

So let's divide by $(\lambda+2)$. Now $-\lambda^3-4\lambda^2-5\lambda-2 = (\lambda+2)(-\lambda^2-2\lambda-1)$

$\downarrow$ roots: -1 (algebraic multiplicity 2)

$=-(\lambda+2)(\lambda+1)^2$

So the eigenvalues are -2 and -1.

To see whether A is diagonalizable, we need to see if $g_{-1}=2$.

Now $g_{-2}=\dim(E_{-2})$, and $E_{-2}=\mathrm{Ker}(A+2I_3) = \mathrm{Ker}\underbrace{\begin{pmatrix}0&-1&1\\2&3&-2\\2&2&-1\end{pmatrix}}$

Clearly this matrix has rank 2, so $g_{-1}=3-2=1$

Since $g_{-1}=1<2=a_{-1}$, A is <u>not</u> diagonalizable.

(d) $A=\begin{pmatrix}-2&2&1\\2&1&2\\1&2&-2\end{pmatrix}$

Since A is symmetric, it is orthogonally diagonalizable.

Eigenvalues: char poly(A) = $\det\begin{pmatrix}-2-\lambda&2&1\\2&1-\lambda&2\\1&2&-2-\lambda\end{pmatrix}$

$$=(-2-\lambda)\cdot\big((1-\lambda)(-2-\lambda)-4\big)-2\cdot\big(2\cdot(-2-\lambda)-2\big)+\big(2\cdot2-(1-\lambda)\big)$$

$$= (-2-\lambda)\cdot(\lambda^2+\lambda-6) - 2\cdot(-2\lambda-6) + (3+\lambda)$$

$$= -\lambda^3-\lambda^2+6\lambda-2\lambda^2-2\lambda+12+4\lambda+12+3+\lambda$$

$$= -\lambda^3-3\lambda^2+9\lambda+27 \quad (*)$$

Observe: $\lambda=3$ is a solution: $-3^3-3\cdot 3^2+9\cdot 3+27$

Next, divide by $\lambda-3$: $-\lambda^3-3\lambda^2+9\lambda+27 = (\lambda-3)\cdot(-\lambda^2-6\lambda-9)$

roots: $\lambda=-3$

(algebraic multiplicity 2)

So the eigenvalues are 3 and -3

Eigenspaces:

$$E_3 = \operatorname{Ker}(A-3I_3) = \operatorname{Ker}\begin{pmatrix}-5 & 2 & 1\\ 2 & -2 & 2\\ 1 & 2 & -5\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}-5 & 2 & 1 & 0\\ 2 & -2 & 2 & 0\\ 1 & 2 & -5 & 0\end{array}\right) \xrightarrow{\mathrm{I}\leftrightarrow\mathrm{III}} \left(\begin{array}{ccc|c}1 & 2 & -5 & 0\\ 2 & -2 & 2 & 0\\ -5 & 2 & 1 & 0\end{array}\right) \xrightarrow[\mathrm{III}\to\mathrm{III}+5\cdot\mathrm{I}]{\mathrm{II}\to\mathrm{II}-2\cdot\mathrm{I}} \left(\begin{array}{ccc|c}1 & 2 & -5 & 0\\ 0 & -6 & 12 & 0\\ 0 & 12 & -24 & 0\end{array}\right)$$

$$\xrightarrow{\mathrm{II}\to\frac{1}{-6}\mathrm{II}} \left(\begin{array}{ccc|c}1 & 2 & -5 & 0\\ 0 & 1 & -2 & 0\\ 0 & 12 & -24 & 0\end{array}\right) \xrightarrow{\mathrm{III}\to\mathrm{III}-12\cdot\mathrm{II}} \left(\begin{array}{ccc|c}1 & 2 & -5 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \xrightarrow{\mathrm{I}\to\mathrm{I}-2\cdot\mathrm{II}} \left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$\hookrightarrow t$

$$\Rightarrow E_3 = \left\{\begin{pmatrix}t\\ 2t\\ t\end{pmatrix} : t\in\mathbb{R}\right\} = \operatorname{Span}\left(\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}\right) \longrightarrow \text{Orthonormal basis: } \frac{1}{\sqrt{6}}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$$

$$E_{-3} = \operatorname{Ker}\begin{pmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 2 & 4 & 2 & 0\\ 1 & 2 & 1 & 0\end{array}\right) \xrightarrow[\mathrm{III}\to\mathrm{III}-\mathrm{I}]{\mathrm{II}\to\mathrm{II}-2\cdot\mathrm{I}} \left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \Rightarrow E_3 = \left\{\begin{pmatrix}-2t-s\\ t\\ s\end{pmatrix} : t,s\in\mathbb{R}\right\}$$

$t \quad s$

$$= \operatorname{Span}\left(\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\right)$$

Now we find an orthonormal basis of E_{-3}. Let $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}$

If $w = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $w^{\perp} = w - (w \cdot u_2) \cdot u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/5 \\ -2/5 \\ 1 \end{pmatrix}$

Finally $u_3 = \frac{w^{\perp}}{\|w^{\perp}\|} = \frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$

$$A = \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{30} & -2/\sqrt{30} & 5/\sqrt{30} \end{pmatrix}$$

2. (a) Note that $S = \operatorname{Ker}(2\ 1\ -1)$, so S is a subspace of $\mathbb{R}^3$.

Since $(2\ 1\ -1)$ has rank 1, by rank-nullity, $\dim(\operatorname{Ker}(2\ 1\ -1)) = 2$.

(b) Let us find an orthonormal basis of S.

Now, a basis for S is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ($\dim(S) = 2$ and these are l.i.).

Gram-Schmidt: $u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $v_2^{\perp} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 1 \\ 1/5 \end{pmatrix}$

$\Rightarrow u_2 = \frac{1}{\sqrt{4+25+1}} \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$

Let $Q = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{30} \\ 0 & 5/\sqrt{30} \\ 2/\sqrt{5} & 1/\sqrt{30} \end{pmatrix}$. The matrix of proj_S is

$$QQ^T = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{30} \\ 0 & 5/\sqrt{30} \\ 2/\sqrt{5} & 1/\sqrt{30} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{30} & 5/\sqrt{30} & 1/\sqrt{30} \end{pmatrix} = \begin{pmatrix} 1/5 + 4/30 & -10/30 & 2/5 - 2/30 \\ -10/30 & 25/30 & 5/30 \\ 2/5 - 2/30 & 5/30 & 4/5 + 1/30 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 5/6 & 1/6 \\ 1/3 & 1/6 & 5/6 \end{pmatrix}.$$

(c) $v^{\parallel} = QQ^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/3 - 2/3 + 3/3 \\ -1/3 + 10/6 + 3/6 \\ 1/3 + 2/6 + 15/6 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 11/6 \\ 19/6 \end{pmatrix}$

$v^{\perp} = v - v^{\parallel} = \begin{pmatrix} 1/3 \\ 1/6 \\ -1/6 \end{pmatrix}.$

3. char poly $(A) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & a-\lambda \end{pmatrix} = \lambda^2 - a\lambda + 1$

Roots: $\lambda = \frac{a \pm \sqrt{a^2-4}}{2}$. This is a single value if and only if $a^2 - 4 = 0$, iff $a = \pm 2$.

If $a=2$, $\lambda = 1$, and $E_1 = \operatorname{Ker} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, so $g_2 = 1 < 2 = a_2$,
so A is not diagonalizable.

If $a=-2$, $\lambda = -1$, and $E_{-1} = \operatorname{Ker} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, so $g_{-2} = 1 < 2 = a_{-2}$
so A is not diagonalizable.

4. (a): $T: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$. Indeed, $\operatorname{Ker}(T) = \{0\}$ but $\operatorname{Im}(T) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \neq \mathbb{R}^2$.

(b): $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, by Exercise 2.

(c) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, since char poly $(A) = \lambda^2 + 1$. This has two distinct eigenvalues over $\mathbb{C}$, so it is diagonalizable over $\mathbb{C}$, but the eigenvalues are complex, so A is not diagonalizable over $\mathbb{R}$.

(d) $A = \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$ Indeed, $\mathrm{Ker}(A) = \mathrm{Span}(e_5, e_6, e_7) \subseteq \mathbb{R}^7$

$\mathrm{Im}(A) = \mathrm{Span}(e_1, e_2, e_3, e_4) \subseteq \mathbb{R}^7$

(e) Consider the nonorthogonal vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$, and take A such that

$A\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $A\cdot\begin{pmatrix}1\\1\end{pmatrix} = 2\cdot\begin{pmatrix}1\\1\end{pmatrix}$. Then $A\cdot\begin{pmatrix}0\\1\end{pmatrix} = A\cdot\left(\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right) = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$

$= \begin{pmatrix}1\\2\end{pmatrix}$

So the matrix $A = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$ works.

5\. (a) False: $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ is orthogonal but not symmetric.

(b) True: recall that $\Sigma a_\lambda \leq$ degree of char poly(A) $= n$, and $g_\lambda \leq a_\lambda$ for all λ, so

$\Sigma g_\lambda \leq \Sigma a_\lambda = n$.

(c) True: since $v \in S$, $\mathrm{proj}_S(v) = v$

$v \in S^\perp$, $\mathrm{proj}_S(v) = 0$ $\Big\} \Rightarrow v = 0$.

(d) True: let $A = \begin{pmatrix}| & & |\\ v_1 & \cdots & v_n \\ | & & |\end{pmatrix}$. Then $A^TA = \begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & \cdots & v_1\cdot v_n \\ \vdots & & & \vdots \\ v_n\cdot v_1 & \cdots & \cdots & v_n \cdot v_n\end{pmatrix} = \begin{pmatrix}\|v_1\|^2 & & 0 \\ & \ddots & \\ 0 & & \|v_n\|^2\end{pmatrix}$.

(e): True: the Jordan normal form of A is either

$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$, $\begin{pmatrix}1&1&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$, $\begin{pmatrix}1&1&&\\&1&1&\\&&1&\\&&&1\end{pmatrix}$, $\begin{pmatrix}1&1&&\\&1&1&\\&&1&1\\&&&1\end{pmatrix}$, or $\begin{pmatrix}1&1&&\\&1&&\\&&1&1\\&&&1\end{pmatrix}$

In each case, $A - I_4 = S^{-1}JS - I_4 = S^{-1}(J-I)S$, so

$(A-I)^4 = S^{-1}\underbrace{(J-I_4)^4}_{\text{zero in every case}}S$

zero in every case: $\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}^4 = \begin{pmatrix}0&1&&\\&0&&\\&&0&\\&&&0\end{pmatrix}^4 = \begin{pmatrix}0&1&&\\&0&1&\\&&0&\\&&&0\end{pmatrix}^4 = \begin{pmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{pmatrix}^4 = \begin{pmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{pmatrix}^4 = \begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&0\end{pmatrix}$

6. Since V^T is invertible, using the hint, $\operatorname{Im}(A) = \operatorname{Im}(U\Sigma)$. Now

$$\operatorname{Im}\left(U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\right) = \operatorname{Span}(\text{first } r \text{ columns of } U) = \operatorname{Span}(u_1, \dots, u_r).$$

Now obviously $u_1, \dots, u_r$ form an orthonormal basis of $\operatorname{Im}(A)$, since they are orthonormal and they span $\operatorname{Im}(A)$.

Next,
$$\begin{aligned}
\operatorname{Ker}(A^T) &= \{v \in \mathbb{R}^n : A^T v = 0\} \\
&\overset{\text{SVD}}{=} \{v \in \mathbb{R}^n : V \Sigma U^T v = 0\} \\
&= \{v \in \mathbb{R}^n : \Sigma U^T v \in \operatorname{Ker}(V)\} \\
&\overset{\operatorname{Ker}(V) = \{0\}}{=} \{v \in \mathbb{R}^n \text{ s.t. } \Sigma U^T v = 0\} \\
&= \left\{v \in \mathbb{R}^n \text{ s.t. } \begin{array}{c} \lambda_1 \cdot (u_1 \cdot v) = 0 \\ \vdots \\ \lambda_r \cdot (u_r \cdot v) = 0 \end{array}\right\} \\
&= \{v \in \mathbb{R}^n : \text{first } r \text{ columns of } U \text{ are orthogonal to } v\} \\
&= \operatorname{Span}(u_1, \dots, u_r)^{\perp}
\end{aligned}$$

Clearly, $u_{r+1}, \dots, u_n$ form an orthonormal basis of this subspace.