Linear Algebra Final 8/5

1. (20 points) For each of the following matrices A, determine whether A is orthogonally diagonalizable, diagonalizable (over $\mathbb{R}$), or neither, justifying your answers. In the case that A is orthogonally diagonalizable, compute its orthogonal diagonalization. (If A is only diagonalizable, you do not need to diagonalize A.)

   (a) $\begin{pmatrix} 4 & 4 \\ -1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} -2 & -1 & 1 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$

   (d) $\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$

2. (20 points) Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z = 2x + y \right\}$.

   (a) Prove that S is a subspace of $\mathbb{R}^3$, and that it has dimension 2.

   (b) Find the matrix of the linear transformation proj_S.

   (c) If $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, compute $v^{||}$ (the orthogonal projection of v onto S), and $v^{\perp}$, the corresponding perpendicular component.

3. (20 points) Find the values of a for which the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$ has only one eigenvalue (of algebraic multiplicity 2). Prove that A is not diagonalizable in each such case.

4. (20 points) Find examples for each of the following. In each case, verify that your example satisfies the conditions given.

   (a) An injective but not surjective linear transformation.

   (b) A matrix which is not diagonalizable over $\mathbb{C}$.

   (c) A matrix diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

(d) A matrix A with $\dim(\text{Ker}(A)) = 3$ and $\dim(\text{Im}(A)) = 4$.

(e) A matrix with two eigenvalues λ and μ (with $\lambda \neq \mu$) such that the eigenspaces E_λ and E_μ are not orthogonal.

5. (20 points) Determine if the following statements are true or false. If they are true, provide a proof. If they are false, provide a counterexample.

   (a) Every orthogonal matrix is orthogonally diagonalizable.

   (b) Let A be an $n \times n$ matrix. Then the sum of the geometric multiplicities of the eigenvalues of A cannot be greater than n

   (c) Let $S \subset \mathbb{R}^n$ be a subspace. Then any vector $v \in \mathbb{R}^n$ such that $v \in S$ and $v \in S^\perp$ must be the zero vector. (Recall that $S^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } w \in S\}$.)

   (d) If A is a matrix whose columns are orthogonal (but not necessarily unit vectors), then $A^T A$ is a diagonal matrix.

   (e) If A is a 4×4 matrix with a single eigenvalue 1 with algebraic multiplicity $a_1 = 4$, then $(A - I)^4 = 0$.

6. (**Extra:** 10 points) Let A be an $m \times n$ matrix (with real entries), and let $A = U\Sigma V^T$ be a Singular Value Decomposition for A. Write $D = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where D is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_r$, all nonzero. Let the columns of U form the vectors $u_1, \ldots, u_n$.
   Prove that $u_1, \ldots, u_r$ form an orthonormal basis of $\text{Im}(A)$, and $u_{r+1}, \ldots, u_n$ form an orthonormal basis of $\text{Ker}(A^T)$. (You may use the following fact without proof: if X and Y are matrices and Y is invertible, then $\text{Im}(XY) = \text{Im}(X)$).