Midterm solutions

1. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow{\text{I}\to -\text{I}} \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow{\text{II}\to\text{II}-2\text{I}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow{\text{II}\to\frac{1}{3}\text{II}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1/3 \end{pmatrix} \xrightarrow{\text{I}\to\text{I}+2\text{II}} \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{pmatrix}$

rank(A) = 2

$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{II}\to\text{II}-\text{I}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{II}\to -\text{II}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{III}\to\text{III}-\text{II}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ rank(B) = 2

b) $AB = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} \xrightarrow{\text{II}\to\text{II}-\text{I}} \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} \xrightarrow{\text{II}\to\frac{1}{4}\text{II}} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{I}\to\text{I}+2\text{II}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \Rightarrow$ rank(AB) = 2

d) Inverse: $\left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}\to\text{II}-\text{I}} \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 4 & -1 & 1 \end{array}\right) \xrightarrow{\text{II}\to\frac{1}{4}\text{II}} \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right) \xrightarrow{\text{I}\to\text{I}+2\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$

$\Rightarrow \quad (AB)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$

2. a) The linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 2y - 2z \\ -x - y + z \end{pmatrix}$

Matrix: $\begin{pmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix}$

Kernel: $\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{\text{I}\to\frac{1}{2}\text{I}} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} & \downarrow & \downarrow \\ & t & s \end{array} \end{array} \Rightarrow \operatorname{Ker}(T) = \left\{ \begin{pmatrix} -t+s \\ t \\ s \end{pmatrix} : t, s \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$

Im(T) : has basis $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

b) The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(v) = -v$ for all $v \in \mathbb{R}^2$.

Matrix: $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Notice that $A^2 = I_2$ so A is its own inverse. Therefore $\operatorname{Ker}(T) = \{0\}$ and Im(T) has basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c) The linear transformation such that $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

Ker (T): $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{II}\to\text{II}-\text{I}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow[\text{III}\to\text{III}-\text{II}]{\text{I}\to\text{I}-\text{II}} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} & & \downarrow \\ & & t \end{array} \end{array} \Rightarrow \operatorname{Ker}(T) = \operatorname{Span}\Big(\underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\text{basis for Ker}(T)}\Big)$.

Im (T): From the rref above we see that $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ form a basis for Im(T).

d) The composition of the transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$

where $T_1\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\y+z\end{pmatrix}$ and $T_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\y\end{pmatrix}$

T_1 has matrix $\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}$. T_2 has matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}$. $T_2\circ T_1$ has matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix} = \begin{pmatrix}1&2&1\\0&1&1\end{pmatrix}$

$\operatorname{Ker}(T_2\circ T_1)$: $\left(\begin{array}{ccc|c}1&2&1&0\\0&1&1&0\end{array}\right) \xrightarrow{I \to I-2II} \begin{array}{c}\left(\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\end{array}\right)\\ \qquad\quad t\end{array} \Rightarrow \operatorname{Ker}(T_2\circ T_1) = \left\{\begin{pmatrix}t\\-t\\t\end{pmatrix} : t\in\mathbb{R}\right\} = \operatorname{Span}\left(\underbrace{\begin{pmatrix}1\\-1\\1\end{pmatrix}}_{\text{basis for the kernel}}\right)$

$\operatorname{Im}(T_2\circ T_1)$: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ since $\operatorname{rank}(T_2\circ T_1) = 2 = \dim(\mathbb{R}^2)$.

3. • $T_1 + T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$(T_1+T_2)(v) = T_1(v) + T_2(v)$

We show that this is linear: for all $v, w \in \mathbb{R}^n$ we have

$$(T_1+T_2)(v+w) = T_1(v+w) + T_2(v+w)$$
$$= T_1(v) + T_1(w) + T_2(v) + T_2(w)$$
$$= T_1(v) + T_2(v) + T_1(w) + T_2(w)$$
$$= (T_1+T_2)(v) + (T_1+T_2)(w).$$

For all $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$ we have

$$(T_1+T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v)$$
$$= \lambda T_1(v) + \lambda \cdot T_2(v)$$
$$= \lambda \cdot (T_1+T_2)(v)$$

Therefore $T_1 + T_2$ is a linear transformation. Its matrix is $\begin{pmatrix}a_{11}+b_{11} & \cdots & a_{1n}+b_{1n}\\ \vdots & & \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn}\end{pmatrix}$

• $cT : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$(cT)(v) = c\cdot T(v)$

$$(cT)(v+w) = c\cdot T(v+w)$$
$$= c\cdot T(v) + c\cdot T(w)$$

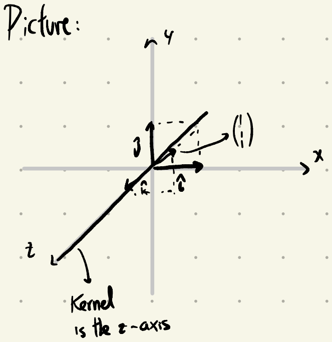
$$= (cT)(v) + (cT)(w)$$

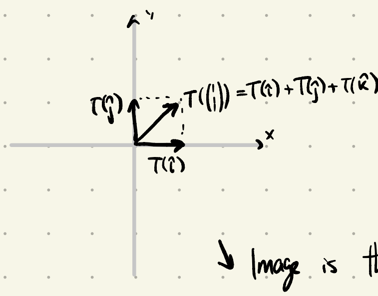
$$(cT)(\lambda v) = c \cdot T(\lambda v)$$
$$= c \cdot \lambda \cdot T(v)$$
$$= \lambda \cdot (cT)(v)$$

Therefore cT is a linear transformation. Its matrix is $\begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$.

4. Consider $T: \mathbb{R}^3 \to \mathbb{R}^2$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Picture:





↘ Image is the whole plane, $\mathbb{R}^2$

5. (a) False: consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The system $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$ is consistent.

(b) True: $\mathrm{Ker}(T_2 \circ T_1) = \{v \in \mathbb{R}^n : (T_2 \circ T_1)(v) = 0\}$

$$= \{v \in \mathbb{R}^n : T_2(T_1(v)) = 0\}$$
$$= \{v \in \mathbb{R}^n : T_1(v) \in \mathrm{Ker}(T_2)\}$$
$$\underset{T_2 \text{ inj}}{=} \{v \in \mathbb{R}^n : T_1(v) = 0\}$$
$$= \mathrm{Ker}(T_1)$$
$$\underset{T_1 \text{ inj}}{=} \{0\}$$

(c) False: consider $T_1 : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ (obviously inj) and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ (obviously surj). However $T_2 \circ T_1 = T_1$ which is not surjective:

$$\dim(\mathrm{Im}(T_1)) = 1 < 2 = \dim(\mathbb{R}^2)$$

(d) True: by rank-nullity, $6 = \dim(\mathrm{Ker}(T)) + \underbrace{\dim(\mathrm{Im}(T))}_{\leq 4}$, so $\dim(\mathrm{Ker}(T)) \geq 2$.

(e) False: consider T_1 given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and T_2 given by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\dim(\mathrm{Im}(T_1)) = \dim(\mathrm{Im}(T_2)) = 2 > 0$.

However $\dim(\mathrm{Im}(T_1+T_2)) = 0$.

$\hookrightarrow$ matrix is $\begin{pmatrix} 1-1 & 0 \\ 0 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6. $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^p$

$AB = 0 \Rightarrow \mathrm{Ker}(B) \supseteq \mathrm{Im}(A) \Rightarrow \dim(\mathrm{Ker}(B)) \geq \dim(\mathrm{Im}(A)) \quad (*)$

Now by rank-nullity, $\dim(\mathrm{Ker}(B)) + \underbrace{\dim(\mathrm{Im}(B))}_{= p \,\to\, T_2 \text{ surj.}} = m$

$\underbrace{\dim(\mathrm{Ker}(A))}_{= 0 \,\to\, T_1 \text{ inj.}} + \dim(\mathrm{Im}(A)) = n$

So $m = \dim(\mathrm{Ker}(B)) + p \underset{(*)}{\geq} \dim(\mathrm{Im}(A)) + p = n + p$.

Equality is achieved iff $\dim(\mathrm{Ker}(B)) = \dim(\mathrm{Im}(A))$ i.e. iff $\mathrm{Ker}(B) = \mathrm{Im}(A)$.