Linear Algebra Midterm

1. (20 points) Let $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$.

   (a) Find the rank of A and the rank of B.

   (b) Compute AB.

   (c) Find the rank of AB.

   (d) Indicate whether AB has an inverse, and compute it if it does.

2. (20 points) For each of the following linear transformations, write down the corresponding matrix, and find bases for its kernel and its image.

   (a) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 2y - 2z \\ -x - y + z \end{pmatrix}$

   (b) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(v) = -v$ for all $v \in \mathbb{R}^2$.

   (c) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

   (d) The composition of the transformations $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ where $T_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \end{pmatrix}$ and $T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$.

3. (20 points) Let T_1, T_2 be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let $c \in \mathbb{R}$ be a scalar. Let also $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ be the matrix associated to T_1 and $B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$ the matrix associated to T_2.

   (a) Define $T_1 + T_2 : \mathbb{R}^n \to \mathbb{R}^m$ as $(T_1 + T_2)(v) = T_1(v) + T_2(v)$ for every $v \in \mathbb{R}^n$. Prove that $T_1 + T_2$ is a linear transformation.

   (b) Write down the matrix for $T_1 + T_2$ in terms of the entries of A and B.

(c) Define $cT_1 : \mathbb{R}^n \to \mathbb{R}^m$ as $(cT_1)(v) = c \cdot T_1(v)$ for every $v \in \mathbb{R}^n$. Prove that cT_1 is a linear transformation.

(d) Write down the matrix for cT_1 in terms of the entries of A.

4. (20 points) Give an example of a surjective linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. Draw your linear transformation in the way we have done it in the course, identifying in your picture: where the basis vectors $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\hat{\mathrm{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ go, the kernel and the image. Explain also how to obtain the image of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in terms of the drawing.

5. (20 points) Determine whether the following statements are true or false. Justify your answers: give a proof if they are true and give a counterexample if they are false.

   (a) If A is an $n \times n$ matrix, but $rank(A) < n$, then the system $Ax = b$ is always inconsistent.

   (b) If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ are injective linear transformations, then $T_2 \circ T_1$ is injective.

   (c) If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ is an injective linear transformation and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ is a surjective linear transformation, then $T_2 \circ T_1$ is surjective.

   (d) A linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^4$ must have $\dim(\mathrm{Ker}(T)) \geq 2$.

   (e) Let T_1 and T_2 be as in question 3. If $\dim(\mathrm{Im}(T_1)) > 0$ and $\dim(\mathrm{Im}(T_2)) > 0$ then $\dim(\mathrm{Im}(T_1 + T_2)) > 0$.

6. (**Extra:** 10 points) Let A be a $m \times n$ matrix corresponding to an injective linear transformation, and let B be a $p \times m$ matrix corresponding to a surjective linear transformation, satisfying that $BA = 0_{p \times n}$ (the RHS is the $p \times n$ matrix with zeros in every entry). Prove that $m \geq n + p$ and identify the condition for when equality holds.