Linear Algebra HW Week 4

1. (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ -3 & -2 & -1 \end{pmatrix}$ and let $\mathcal{B}$ be the basis consisting of $v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

   (a) Compute $S_{\mathcal{B}\to\mathcal{C}}$ and $S_{\mathcal{C}\to\mathcal{B}}$.

   (b) Compute $[v]_{\mathcal{B}}$ where $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   (c) Compute v given that $[v]_{\mathcal{B}} = \begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix}$

   (d) Compute the matrix of T with respect to the basis $\mathcal{B}$ and compute $[T(v)]_{\mathcal{B}}$, where v is the vector in part (c).

2. (40 points) Find the eigenvalues and eigenspaces of the following matrices. Indicate the corresponding algebraic and geometric multiplicities. Indicate whether the matrix is diagonalizable, and if possible, diagonalize the matrix.

   (a) $\begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -6 & -3 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} -1 & 4 & 1 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 3 & 0 & 0 \\ -11 & -4 & 2 \\ -25 & -10 & 4 \end{pmatrix}$

3. (20 points) Determine if the following statements are true or false. If they are true, provide a proof. If they are false, provide a counterexample.

   (a) If A is a square matrix which is not invertible, then 0 is an eigenvalue of A.

   (b) If A is a diagonalizable matrix, then all of its eigenvalues must be different.

(c) If A is a square matrix of rank 1 and A is similar to B, then B must have rank 1.

(d) If A and B are square matrices of the same rank, then A and B must be similar.

4. (20 points) Prove the following statements.

(a) Let λ be a fixed real number and let A and B be square $n \times n$ matrices such that $A \sim B$. Then $A - \lambda I_n$ is similar to $B - \lambda I_n$.

(b) Let $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \ddots & 0 \\ \vdots & & & \vdots \\ a_{n-1,1} & & a_{n-1,n-1} & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ be a lower triangular matrix. Then its eigenvalues are $a_{11}, ..., a_{nn}$.

(c) The trace $\mathrm{tr}(A)$ of an arbitrary square matrix A is defined as the sum of its diagonal entries. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any square $n \times n$ matrices A and B. Deduce that if A_1 and A_2 are similar matrices, then $\mathrm{tr}(A_1) = \mathrm{tr}(A_2)$.

(d) Use the previous part to show that the characteristic polynomial of a diagonalizable 2×2 matrix is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$.