Linear Algebra HW Week 3

1. Compute the determinants of the following matrices. First, use the rref definition of the determinant, then use Laplacian expansion along the second row (pay attention to the signs in Theorem 3 from Lecture 10).

   (a) $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 \end{pmatrix}$

2. Consider the matrices matrices $A = \begin{pmatrix} 3 & -1 \\ 2 & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

   (a) Compute $\det(AB)$

   (b) Compute $\det(A^2 B^{-1} A^T)$

   (c) Compute $\det((B^{-1})^T)$

   (d) Compute $\det(C^{-1}ABC)$, where C is an invertible 2×2 whose entries you forgot.

3. Use Laplacian expansion to prove the following equalities.

   (a) Adding two columns:

$$\det\begin{pmatrix} a_{11}+a'_{11} & a_{12} & \dots & a_{1n} \\ a_{21}+a'_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1}+a'_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a'_{11} & a_{12} & \dots & a_{1n} \\ a'_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a'_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

   (b) Multiplying a column by a scalar $c \in \mathbb{R}$:

$$\det\begin{pmatrix} ca_{11} & a_{12} & \dots & a_{1n} \\ ca_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ ca_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = c\det\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

   (c) Swapping the first two columns: (hint: expand along the second column on the LHS and along the right column on the RHS)

$$\det\begin{pmatrix} a_{12} & a_{11} & \dots & a_{1n} \\ a_{22} & a_{21} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n2} & a_{n1} & \dots & a_{nn} \end{pmatrix} = -\det\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$