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NOTES ON CONDENSED MATHEMATICS

Abstract. These are notes based on condensed mathematics learning seminar at Princeton University organized by Vadim Vologodsky and Bogdan Zavyalov. The goal is to understand the notion of solid A-module. The main application to discuss is to the formalism of six-functors in the coherent setting. Unless otherwise specified, all the results below are due to Dustin Clausen and Peter Scholze and the references within [SCa] and [SCc]. Talks are given (in order) by Bogdan Zavyalov, Steven Jin, Longke Tang, Vadim Vologodsky, Sally Gilles, Siqing Zhang, Juan Esteban Rodriguez Camargo, Dmitry Kubrak, Hanlin Cai and Tasos Moulinos.

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[1] Notes were taken neither fully-faithfully nor essential-surjectively by Hanlin Cai, who claimed all the errors and mistakes.

1. Introduction

The main goal of condensed mathematics is to do homological algebra with topological structures. However, there are some problems.

(1) The category of abelian topological groups is not an abelian category. For example, let $\mathbb{R}^{\text{disc}}$ be the real numbers endowed with the discrete topology and $\mathbb{R}^{\text{eucl}}$ be the real numbers endowed with the natural Euclidean topology and we consider the continuous morphism

$$f : \mathbb{R}^{\text{disc}} \to \mathbb{R}^{\text{eucl}}$$

by $x \mapsto x$. Clearly we have $\ker(f) \simeq 0$ and $\operatorname{coker}(f) \simeq 0$. However this "identity" is not an isomorphism since the topologies is different. So one of the axioms (AB2) for the abelian category is violated.

The second example in this direction is to consider topological groups. If G is a Hausdorff topological group and H is a topological normal subgroup of G then taking the quotient G/H is not necessary to be Hausdorff unless H is closed. Moreover, if one consider a short exact sequence of continuous G-modules, then it does not in general give a long exact sequence of continuous group cohomology groups.

(2) There is no good quasicoherent sheaves theory for "analytic spaces" as opposed to the classical setting of schemes. For example, if X is a scheme (or just an algebraic variety), then we have the following embedding of categories:

$$\operatorname{Vect}(X) \hookrightarrow \operatorname{Coh}(X) \hookrightarrow \operatorname{Qcoh}(X) \hookrightarrow \mathcal{O}_X - \operatorname{Mod}.$$

where the first embedding is by adding kernels and cokernels and the second embedding is by adding all the colimits. In the affine case where $X = \operatorname{Spec}(A)$, the above is just:

$$\operatorname{Mod}_A^{\text{f.g.proj}} \hookrightarrow \operatorname{Mod}_A^{\text{f.g.}} \hookrightarrow \operatorname{Mod}_A \hookrightarrow \mathcal{O}_{\operatorname{Spec}(A)} - \operatorname{Mod}$$

However in the case where $\mathfrak{X}$ is either a complex manifold or a rigid analytic space over $\mathbb{Q}_p$ over a "nice" formal scheme over $\mathbb{Z}_p$, there is no good theory like this. For instance, if we let $\mathfrak{X} = \operatorname{Spf}(\mathbb{Z}_p)$, then it is clear that the first two categories should be $\operatorname{Mod}_{\mathbb{Z}_p}^{\text{f.g.free}}$ and $\operatorname{Mod}_{\mathbb{Z}_p}^{\text{f.g.}}$. But the naive attempt $\operatorname{Mod}_{\mathbb{Z}_p}^{\text{comp}}$ (the p-adically complete $\mathbb{Z}_p$-modules) for the quasicoherent sheaves won't work as this category is not abelian[1].

(3) We want to have a 6-functor formalism in the (quasi)coherent setting, but there is not even coherent lower shriek functor even for schemes. Recall that in the topological setting, given an open embedding $j : U \hookrightarrow X$ of topological spaces, one has a restriction functor on sheaf of sets:

$$j^* : \operatorname{Shv}(X) \to \operatorname{Shv}(U)$$

by $F \mapsto (F\mid_U : V \mapsto F(V))$ for every $V \subset U$. And it admits a left adjoint "extension by zero":

$$j_! : \operatorname{Shv}(U) \hookrightarrow \operatorname{Shv}(X)$$

[1]It is easy to see that the cokernel doesn't exist, since it might not be complete. One can consider $M = \mathbb{Z}_p\langle x\rangle$, $M' = \mathbb{Z}_p[[px]]$ and $M'' = \mathbb{Z}_p\langle px\rangle$. Then M/M'' is not p-adically complete since it is not p-adically separated and its p-adic completion is M/M'.

by sending F to the sheafification of the presheaf $(V \mapsto F(V))$ if $V \subset U$ and $(V \mapsto \varnothing)$ otherwise. We have the bijection

$$\mathrm{Hom}_X(j_!F, G) \simeq \mathrm{Hom}_U(F, j^*G)$$

for every $F \in \mathrm{Shv}(U)$ and $G \in \mathrm{Shv}(X)$.

However, there is a lack of such theory in algebraic geometry for (quasi)coherent sheaves. One can see this already on the level of affine schemes. Take $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(A_f)$ then we have

$$j^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(U).$$

We may identify the above functor as

$$j^* : \mathrm{Mod}_A \to \mathrm{Mod}_{A_f}$$

by sending M to $M \otimes_A A_f$. But j^* doesn't commute with infinite products, so it fails to have a left adjoint. Notice that if one endow A with the discrete topology then the natural topology on some infinite product of A is not discrete. And to solve this issue, you want $\otimes_A A_f$ to be "topological" in sense, i.e. to have a "good" notion of "complete tensor".

We start by looking a little bit into the first problem. To solve the first problem we want to define a category Cond(Ab) which is called the category of condensed abelian groups to have the following properties:

(1) The category of "nice" topological spaces should embed into Cond(Ab).

(2) Cond(Ab) should behave like the category of abelian groups:

(2.1) Cond(Ab) is an abelian category, i.e. an additive category where kernels and cokernels exist, and the natural map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism for every morphism.

(2.2) Taking infinite products are exact, i.e. if one has infinite many short exact sequences

$$0 \to M_i \to M_i' \to M_i'' \to 0$$

then they induce a short exact sequence

$$0 \to \prod M_i \to \prod M_i' \to \prod M_i'' \to 0.$$

The idea is to realize Cond(Ab) as the abelian sheaves on some Grothendieck topology, then (1) and (2.1) will follow immediately. But we will need Stone-Čech compactification to prove (2.2). Before giving the definition, we recall with the following example[2].

Example 1.1. We denote CHaus by the category of compact Hausdorff spaces, with continuous maps between them and with covers given by finite families of jointly surjective maps.

Remark 1.2. Note that finite fibre products exist in CHaus. This due to the fact that any closed subspace of a compact Hausdorff space is compact and the diagonal of the self product of some Hausdorff space is closed. Then one can realize the fibre product of $K_i \to K \in \mathrm{CHaus}$ as the preimage of the diagonal under the natural map $\prod K_i \to \prod K$.

[2]For simplicity, here we work in a suitable Grothendieck universe to avoid some set theoretical issues. We will deal with the set theoretical issues in the next section.

Remark 1.3. The Grothendieck topology on CHaus is generated by the following two types of coverings:

(1) Finite colletion of $K_i \to K$ such that $\coprod K_i \to K$ is an isomorphism.
(2) $K' \to K$ where the map is surjective.

Now we give a definition of condensed sets(/groups/rings/. . .).

Definition 1.4. A condensed set/group/ring/. . . is a sheaf of set/group/ring/. . . on CHaus.

Example 1.5. Let X be any topological space. Then $\underline{X} : K \mapsto \mathrm{Hom}_{\mathrm{Top}}(K, X)$ is a condensed set[3]. By Remark 1.2 and Remark 1.3, to show that $\underline{X}$ is actually a sheaf, we have to verify that $\underline{X}(\varnothing) = *$ (which is clear) and the following two conditions:

(1) The natural map $\underline{X}(K_1 \sqcup K_2) \to \underline{X}(K_1) \times \underline{X}(K_2)$ is a bijection for any $K_1, K_2 \in$ CHaus.
(2) For any surjection $K' \to K$, $\underline{X}(K)$ is the equalizer of the two maps from $\underline{X}(K')$ to $\underline{X}(K' \times_K K')$.

Clearly (1) is satisfied; as for (2), it follows from the fact that any continuous surjection between compact Hausdorff spaces is a quotient map, so any map $f : K \to X$ determined by $g : K' \to K$ and $h : K' \to X$ with $h = f \circ g$ is automatically continuous.

Remark 1.6. We will see later that the category of compactly generated topological spaces "Yoneda" embedds into condensed sets. Moreover, the qcqs objects in the category of condensed sets are exactly represented by compact Hausdorff spaces.

We end this section by returning to the first example in the first problem, the "identity" map $f : \mathbb{R}^{\mathrm{disc}} \to \mathbb{R}^{\mathrm{eucl}}$. Now we view f as a map of sheaves on CHaus.

Example 1.7. The map $f : \underline{\mathbb{R}}^{\mathrm{disc}} \to \underline{\mathbb{R}}^{\mathrm{eucl}}$ is not an isomorphism in Cond(Set). For a compact Hausdorff space K, we have $\underline{\mathbb{R}}^{\mathrm{disc}}(K) = \{\text{locally constant maps } K \to \mathbb{R}^{\mathrm{eucl}}\}$ and $\underline{\mathbb{R}}^{\mathrm{eucl}}(K) = \{\text{continuous maps } K \to \mathbb{R}^{\mathrm{eucl}}\}$. Therefore f is injective with a nontrivial cokernel $\mathrm{coker}(f)$. Even though $\mathrm{coker}(f)(*) = 0$ but $\mathrm{coker}(f) \neq 0$. In fact, we will see later that if K is a profinite set, then

$$\mathrm{coker}(f)(K) = \{\text{continuous maps } K \to \mathbb{R}^{\mathrm{eucl}}\}/\{\text{locally constant maps } K \to \mathbb{R}^{\mathrm{eucl}}\}.$$

[3]Rigorously speaking, this is not true if X is not $T1$ or we do not restrict cardinal. See Remark 3.10.

2. Condensed Objects

We will give the definition(s) of condensed objects in this section. In particular, we will see that the category of condensed abelian groups behaves like the category of abelian groups, i.e. it is not only an abelian category but also satisfies all the Grothendieck's AB axioms, c.f. [Sta, 079B]. We first recall the notion of the pro-objects of a category.

Definition 2.1. Let $\mathcal{C}$ be an essentially small category which admits all finite limits. Let $\mathrm{Pro}(\mathcal{C})$ be the opposite of the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ spanned by left exact functors.

Remark 2.2. The first obersevation is that $\mathrm{Pro}(\mathcal{C})$ admits cofiltered limits. Note that $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ admits filtered colimits and filtered colimits commute with finite limits. Hence $\mathrm{Pro}(\mathcal{C})$ is closed under filtered colimits.

Remark 2.3. We remark here that this construction is the same as Grothendieck is original construction in [Gro60]. In [Gro60], $\mathrm{Pro}(\mathcal{C})$ is taken to be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ which are cofiltered limit of the corepresentable functors under (the opposite) Yoneda embedding. In fact, for any $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$, it has a canonical presentation:

$$\operatorname*{colim}_{(C,\eta)\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}}(C, -)$$

where $\mathcal{A}$ is the "Grothendieck's construction of F" which is the category consisting objects (C, η) with $C \in \mathcal{C}$ and $\eta \in F(C)$; and morphisms

$$\mathrm{Hom}_{\mathcal{A}}((C, \eta), (C', \eta')) = \{f \in \mathrm{Hom}_{\mathcal{C}}(C', C) \text{ such that } F(f)(\eta') = \eta\}.$$

Moreover if F is a left exact functor (which preserves finite limits), $\mathcal{A}$ is filtered, i.e. every finite diagram in $\mathcal{A}$ has a cocone.

Proposition 2.4. $\mathrm{Pro}(\mathcal{C})$ *admits finite limits. Morevoer it has the following property if* $\mathcal{C}$ *has:*

(1) Every morphism $X \to Z$ *factors as* $X \to Y \to Z$ *where* $X \to Y$ *is an effective epimorphism and* $Y \to Z$ *is a monomorphism.*

(2) Finite (disjoint) coproducts exist.

Proof. Easy exercises. □

Remark 2.5. $\mathrm{Pro}(\mathcal{C})$ has the following universal property: for any category $\mathcal{D}$ admits cofiltered limits and any functor $\mathcal{C} \to \mathcal{D}$, there exists a essentially unique functor $\mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ preserving cofiltered limits, i.e. the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by functors preserving cofiltered limits, where the equivalence is induced by the embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$.

Remark 2.6. One can think of $\mathrm{Pro}(\mathcal{C})$ to be the category with objects $\{C_i\}$ indexed by a cofiltered category and with morphisms

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\{C_i\}, \{D_j\}) = \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathcal{C}}(C_i, D_j).$$

We are in particular interested in the situation of profinite sets, i.e. Pro(FinSet). However, the category of profinite sets is very big, so we have to be careful with set theoretical issues. Hence we start with a fixed uncountable strong limit cardinal κ, i.e. for all $\lambda < \kappa$ we have $2^\lambda < \kappa$. Note that there are enough strong limit cardinals. Let $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for a successor ordinal and $\beth_\alpha = \bigcup_{\beta<\alpha} \beth_\beta$ for limit ordinal. Then for any limit ordinal α, $\beth_\alpha$ is an uncountable strong limit cardinal. Now we definie κ-small proétale site of a point.

Definition 2.7. Let $*_{\kappa-\text{proét}}$ be the κ-small proétale site of a point is the category of κ-small profinite sets with covers given by finite families of jointly surjective maps.

The first observation is the following:

Proposition 2.8. *The category $*_{\kappa-\text{proét}}$ is equivalent to the category of κ-small compact Hausdorff totally disconnected topological spaces.*

Proof. Endow finite sets with discrete topology, then we clearly have a fully faithful functor from the former category to the latter. To show essentially surjectivity, fix X to be some κ-small compact Hausdorff totally disconnected. We let $\mathcal{P}$ be the partially ordered set of finite nonempty clopen decomposition of X ordered by refinement. Then there is a natural map from X to for every $P \in \mathcal{P}$ by sending every point to its component. Hence we have a continuous map

$$X \to \lim_{\mathcal{P}} P$$

Then compactness implies surjective and the assumption of totally disconnected on compact Hausdorff implies injective. □

Now we give the definition of condensed objects.

Definition 2.9. Let $\mathcal{C}$ be an essentially small category with finite limits. Then the category of κ-small condensed objects in $\mathcal{C}$ is the category of sheaves on $*_{\kappa-\text{proét}}$ with values in $\mathcal{C}$, denoted as $\text{Cond}_\kappa(\mathcal{C})$.

Note that the definition we give above is different from the Definition 1.4. However, these two definitions are equivalent. In fact, we can give another equivalent description.

Proposition 2.10. *Let $\mathcal{C}$ be an essentially small category with finite limits. Then the following categories are equivalent:*

(1) The category of sheaves on $*_{\kappa-\text{proét}}$ with values in $\mathcal{C}$.
(2) The category of sheaves on CHaus_κ with values in $\mathcal{C}$.
(3) The category of sheaves on κ-small extremally disconnected compact Hausdorff spaces with covers given by finite families of jointly surjective maps.[4]

Before giving the proof, we first recall a few things from general topology.

Construction 2.11. Let X be a topological space and $\mathcal{I}$ be the isomorphism classes of $f_i : X \to Y_i$ where Y_i is compact Hausdorff and $f(X) \subset Y_i$ is dense. Denote βX as the closure of the image of $X \to \prod Y_i$. We call βX the Stone-Čech compactification of X. It is not hard to see that βX

[4]We should be a little careful here by what sheaves mean as we will see later that (fibre) products do not exist for extremally disconnected spaces. Hence we just define sheaves to be functors on the opposite category that send empty set to singleton and preserve finite products.

is compact Hausdorff and is universal among all the maps from X to some objects in CHaus with dense image in its closure. In fact, it is the left adjoint of the inclusion CHaus $\hookrightarrow$ Top. We refer the reader to [Sta, 0908] and [nA, compactum] for more properties of this functor.

Example 2.12. We give a few examples.

(1) If X is compact Hausdorff, then $\beta X = X$.

(2) Let $X = \mathbb{N}$ endowed with the discrete topology, then $\beta\mathbb{N}$ is the set of all ultrafilters on $\mathbb{N}$ endowed with the Stone topology. In fact one can see this by verifying the universal property. Let $S \in$ CHaus and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, then S being compact Hausdorff gives a unique point $u = \lim \mathcal{U}$. Thus we have a map $\beta\mathbb{N} \to S$.

(3) Let $X = \mathbb{A}^1_{\mathbb{R}}$, then $\beta\mathbb{A}^1_{\mathbb{R}}$ is the maximal spectrum of the ring $C^b(\mathbb{A}^1_{\mathbb{R}}, \mathbb{R})$ of bounded continuous real functions on $\mathbb{A}^1_{\mathbb{R}}$. One easily check this by (1) and the universal property (pulling back functions gives the functoriality of $C^b(-, \mathbb{R})$). This examples shows that even in the simple cases, it is hard to describe βX explicitly.

The Stone-Čech compactification comes closely with the concept of extremally disconnected spaces.

Definition 2.13. A compact Hausdroff space S is called extremally disconnected if any surjection $S' \to S$ from a compact Hausdroff space splits.

Remark 2.14. Notice that the closure of every open subspace is open. For every U open in S, the section of $(X \backslash U) \sqcup \overline{U} \to X$ will give $\overline{U}$ open. In fact, the converse is also true, c.f. [Sta, 08YN].

Remark 2.15 (Warning). One immediate consequence of Remark 2.14 is extremally disconnected spaces are totally disconnected. However, as oppose to totally disconnected spaces, product of extremally disconnected spaces are rarely extremally disconnected. In fact, one can look at $\beta S \times \beta S$ for any infinite discrete S. Then the closure of $\{(s, s) \in S \times S\}$ is the diagonal which cannot be open since the space is infinite. Since can be generalized to show that the product of any two infinite extremally disconnected spaces is never extremally disconnected.

Remark 2.16. One can also show that extremally disconnected spaces are exactly the projective objects in CHaus, i.e. for any extremally disconnected space S and any surjection $Y \to Z$, there exists a lift for any $S \to Z$. And there are enough projectives in CHaus, i.e. for every $X \in$ CHaus, there exists some extremally disconnected space S such that S surjects onto X. In fact, one can take S to be βX^{disc} and the map comes from the factorization $X^{\text{disc}} \to \beta X^{\text{disc}} \to X$. Then one can check that this map is surjective, c.f. [Sta, 090D]. This also shows that any extremally disconnected space arises from retracting βX for some discrete X.

Now we are ready to prove Proposition 2.10.

Proof of Proposition 2.10: (1) $\Leftrightarrow$ (2) : It suffices to show that for any sheaf on CHaus_κ, its value is uniquely determined by its restriction to $*_{\kappa-\text{proét}}$. Note that for any $S \in \text{CHaus}_\kappa$, we have a continuous surjective map $\beta S^{\text{disc}} \to S$. Then we win by descent.

(2) $\Leftrightarrow$ (3) : Again we have to verify that any sheave F on $*_{\kappa-\text{proét}}$ determined uniquely by restriction. Let $S \in *_{\kappa-\text{proét}}$. The only part in the above proof does not go through is the fibre

product $T = \beta S^{\text{disc}} \times_S \beta S^{\text{disc}}$ might note be extremally disconnected. Hence we apply the Stone-Čech compactification again and get $T' = \beta T^{\text{disc}}$. Now T' is extremally disconnected and enjoy two maps to βS^{disc}. Since $T' \to T$ is surjective, by sheaf condition $F(T) \to F(T')$ is injective, thus $F(S)$ is a equalizer determined by $F(\beta S^{\text{disc}})$ to $F(T')$. □

The following theorem tells us that the category of condensed abelian groups behaves exactly like the category of abelian groups. This gives us a hint why condensed mathematics is the "correct" way to do homological algebra with topological structures.

Theorem 2.17. *The category of κ-small condensed abelian groups* $\text{Cond}_\kappa(\text{Ab})$ *is an abelian category and satisfies all the Grothendieck's AB axioms. Moreover, it is compactly generated and we can take generators to be projective and compact.*

We first recall a few notions for latter use.

Definition 2.18. See [Lur06, Section 5] for details.

(1) An ∞-category $\mathcal{C}$ is called κ-accessible for some regular cardinal κ, there is a small ∞-category $\mathcal{C}_0$ and an equivalence

$$\text{Ind}_\kappa(\mathcal{C}_0) \to \mathcal{C}.$$

We call an ∞-category accessible if it is κ-accessible for some regular κ. A functor between two κ-accessible ∞-categories is called κ-accessible if it preserves κ-filtered colimits.

(2) An ∞-category is presentable if it is accessible and admits small colimits.

(3) An ∞-category is κ-compactly generated if it is κ-accessible and presentable.

We will drop κ if $\kappa = \aleph_0$.

Remark 2.19. [Lur06, Proposition 5.5.7.8] shows that we can arrange κ-compactly generated $\mathcal{C}$ to be generated under colimits by κ-compact objects, which explains the name of the definition.

Remark 2.20. One benefit of using the first condition in Proposition 2.10 is that $(*_{\kappa-\text{proét}})^{\text{op}} = \text{Ind}(\text{FinSet}^{\text{op}})$ which is presentable. Hence if $\mathcal{C}$ is presentable, e.g. Set or Ab, then $\text{Cond}_\kappa(\mathcal{C})$ can be described as the accessible functors from $(*_{\kappa-\text{proét}})^{\text{op}}$ to $\mathcal{C}$ satisfying sheaf conditions.

Definition 2.21. Let $\mathcal{C}$ be an additive category. Then there are some extra axioms one can assume:

(1) (AB1) Kernel and Cokernel exist.
(2) (AB2) The natural map coim $\to$ im is an isomorphism.
(3) (AB3) All colimits exist.
(4) (AB3*) All limits exist.
(5) (AB4) Taking arbitrary coproduct is exact.
(6) (AB4*) Taking arbitrary product is exact.
(7) (AB5) Taking filtered colimit is exact.

(8) (AB6) Fro any index set J and filtered categories I_j, $j \in J$, with functors $i \mapsto M_i$ from I_j to κ-small condensed abelian groups, The following natural map is an isomorphism:

$$\operatorname*{colim}_{(i_j \in I_j)} \prod_j M_{i_j} \to \prod_j \operatorname*{colim}_{(i_j \in I_j)} M_{i_j}$$

If (AB1) and (AB2) are satisfied, then $\mathcal{C}$ is called an abelian category. Then rest is called Grothendieck's AB axioms.

Proof of Theorem 2.17. Recall the general fact that the category of sheaves of abelian groups on any site satisfies (AB1),(AB2),(AB3),(AB3*). As for the rest of the axioms, they are all true in the category of abelain groups, hence it suffices to show the following claim: all the colimits and limits in $\mathrm{Cond}_\kappa(\mathrm{Ab})$ can be computed objectwise on extremally disconnected spaces. Since all limits and colimits commutes with finite products in Ab, taking the limits and colimits on the presheaf level is the same as the sheaf level on extremally disconnected spaces.

It remains to show the last statement. Since the forgetful functor $\mathrm{Cond}_\kappa(\mathrm{Ab}) \to \mathrm{Cond}_\kappa(\mathrm{Set})$ preserves limit, it admits a left adjoint $T \to \mathbb{Z}[T]$ by adjunction functor theorem, where $\mathbb{Z}[T]$ is the sheafification of the presheaf $S \mapsto \mathbb{Z}[T(S)]$. This is verified on the presheaf level and use the fact that sheafification is left adjoint to the forgetful functor from sheaves to presheaves. Then for any k-small extremally disconnected space S and $M \in \mathrm{Cond}_\kappa(\mathrm{Ab})$ we have $\mathrm{Hom}_{\mathrm{Cond}_\kappa(Ab)}(\mathbb{Z}[\underline{S}], M) \simeq M(S)$ by adjunction and Yoneda lemma. Since S is extremally disconnected, $M \mapsto M(S)$ preserves limits, colimits and (effective) epimorphisms. Hence $\mathbb{Z}[\underline{S}]$ is compact and projective in $\mathrm{Cond}_\kappa(\mathrm{Ab})$ for any S extremally disconnected. To see they are generators, let $M \in \mathrm{Cond}_\kappa(\mathrm{Ab})$ and $\mathcal{I}$ to the set of all subobjects of M admitting a surjection from $\bigoplus \mathbb{Z}[S_\alpha]$. By Zorn's lemma, there exists a maximal one M'. Suppose by contradiction, if $M/M' \neq 0$ then there is a nonzero map from $\mathbb{Z}[\underline{S}] \to M/M'$. Then by the projectivity of $\mathbb{Z}[\underline{S}]$, this map can be lifted to M, hence contradiction. □

Remark 2.22. Tensor products exists in $\mathrm{Cond}_\kappa(\mathrm{Ab})$. For any $M, N \in \mathrm{Cond}_\kappa(\mathrm{Ab})$, $M \otimes N$ is the sheafification of $S \mapsto M(S) \otimes N(S)$. For any $T \in \mathrm{Cond}_\kappa(\mathrm{Set})$, since $\mathbb{Z}[T]$ is free, it is flat. Moreover, it is easy to check that $\mathbb{Z}[T_1 \times T_2] \simeq \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.

Remark 2.23. Internal Hom also exists. One can either see this by using adjunction functor theorem, or construct this concretely by setting

$$\underline{\mathrm{Hom}}(M, N)(S) = \mathrm{Hom}(M \otimes \mathbb{Z}[\underline{S}], N)$$

for any extremally disconneted S.

3. Yoneda Embedding

We take a closer look into condensed sets. One shocking property (as explained in [SCa]) is that $\mathrm{Cond}_\kappa(\mathrm{Set})$ contains "enough" spaces. Recall the following definition. From now, for any $X, Y \in \mathrm{Top}$, we use $C(X,Y)$ to denote $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$, i.e. the continuous morphisms between X and Y.

Definition 3.1. A topological space X is called κ-small compactly generated if it can be realized as a quotient of $\bigsqcup_{S\to X} S$ with $S \in \mathrm{CHaus}_\kappa$. Or equivalently, for any $Y \in \mathrm{Top}_\kappa$ the map $X \to Y$ is continuous iff the composition $S \to X \to Y$ is continuous for all possible $S \in \mathrm{CHaus}_\kappa$.

Remark 3.2. Note that the inclusion from κ-small compactly generated spaces to Top_κ admits right adjoint $X \mapsto X^{\kappa-\mathrm{cg}}$ has the underlying set X and the quotient topology $\bigsqcup_{C(S,X)} S \to X$ for all $S \in \mathrm{CHaus}_\kappa$.

Proposition 3.3. *The "Yoneda embedding"*

$$\mathrm{Top} \to \mathrm{Cond}_\kappa(\mathrm{Set})$$

is faithful; and fully faithful when restrict to the subcategory κ-small compactly generated spaces.

Moreover, this embedding admits a left adjoint $S \to S(*)_{\mathrm{top}}$. $S(*)_{\mathrm{top}}$ has the underlying set $S(*)$ endowed with the quotient topology given by $\bigsqcup_{S(T)} T \to S(*)$ for all $T \in *_{\kappa-\mathrm{pro\acute{e}t}}$.

Proof. We have already seen this "Yoneda embedding" is well-defined in Example 1.5. To show (fully) faithful, it suffices to prove the adjunction

$$C(S(*)_{\mathrm{top}}, X) \simeq \mathrm{Hom}_{\mathrm{Cond}_\kappa(\mathrm{Set})}(S, \underline{X}).$$

Endcoding the information on the left, these are the maps from the set $S(*)$ to X such that the composition $T \to S(*) \to X$ is continuous for all $S(T)$ and $T \in *_{\kappa-\mathrm{pro\acute{e}t}}$. Then by an easy Yoneda yoga, this corresponds exactly to the right hand side. □

Remark 3.4. When we restrict the left adjoint to some representable sheaf $\underline{X}$, we have $\underline{X}(*)_{\mathrm{top}} \simeq X^{\kappa-\mathrm{cg}}$.

Now we want to remove the restriction on cardinal. The idea is to take "the colimit" with respect to κ. To do this rigorously, we need a fully faithfulness result on enlarging cardinals.

Proposition 3.5. *Let $\mathcal{C}$ be a essentially small, presentable category with finite limits. For $\kappa' > \kappa$ be uncountable strong limit cardinals. Then the inclusion $i : \mathrm{CHaus}_\kappa \to \mathrm{CHaus}_{\kappa'}$ induces an adjoint pair of topoi*

$$(i^*, i_*) : \mathrm{Cond}_\kappa(\mathcal{C}) \to \mathrm{Cond}_{\kappa'}(\mathcal{C}).$$

The functor i^* is fully faithful and commutes with all colimits.

Proof. The existence and adjointness of (i^*, i_*) is by general nonsense. Hence we just need to prove fully faithfulness. Note that for some $T \in \mathrm{Cond}_\kappa(\mathcal{C})$, general topos theory tells us that $i^*(T)$ is the sheafification of the presheaf $S' \mapsto \mathrm{colim}_{S'\to S} T(S)$ where S runs through all the κ-small profinite sets. We restrict to the extremally disconnected spaces to avoid sheafification issue. Now we can view i^* as the left Kan extension of the fully faithful inclusion from κ-small extremally disconnected spaces to κ'-small extremally disconnected spaces. Then by [nA, Kan extension, Proposition 4.1] we win. □

Remark 3.6. If $\mathcal{C}$ admits a fully faithful conservative functor to Set, then i^* commutes with all λ-small limits with $\lambda = \mathrm{cf}(\kappa)$, e.g. forgetful functor for Ab, Group etc. Since the functor is fully faithful and conservative, it suffices to prove for Set. Recall the general fact that for any regular cardinal μ, μ filtered colimits commute with μ-small limits in Set. It suffices to show that for any κ'-small extremally disconnected space S', the system of κ-small extremally disconnected space S with a map $S' \to S$ is λ-filtered. Suppose there is a λ-small index category I with compatible maps $S' \to S_i$ with S_i extremally disconnected. Since $\mathrm{cf}(\kappa) = \lambda$, we have $\sup_i |S_i| = \nu < \kappa$ and therefore $|\prod_i S_i| \leq \nu^\lambda < 2^{\nu\times\lambda} < \kappa$. Hence $\lim_i S_i$ is κ-small. Thus we can take $\widetilde{S} = \beta(\lim_i S_i)^{\mathrm{disc}}$ and $\widetilde{S}$ is κ-small surjects onto $\lim_i S_i$. Since S' is extremally disconnected, we get a lift map $S' \to \widetilde{S}$ that is compatible with all the maps.

Definition 3.7. Let $\mathcal{C}$ be a essentially small, presentable category with finite limits. The category $\mathrm{Cond}(\mathcal{C})$ of condensed objects in $\mathcal{C}$ is the filtered colimit of $\mathrm{Cond}_\kappa(\mathcal{C})$ along the uncountable strong limit cardinals.

Remark 3.8. Note that $\mathrm{Cond}(\mathcal{C})$ is not a topos on some site anymore but it enjoy many similar properties. In fact, one should think of this category as tuple (κ, S_κ) with $S_\kappa \in \mathrm{Cond}_\kappa(\mathcal{C})$. And morphisms are defined by Proposition 3.5. Hence even $\mathrm{Cond}(\mathcal{C})$ may not be small when $\mathcal{C}$ is small, it is locally small as long as $\mathcal{C}$ is.

We want to get a similar result as Proposition 3.3 for Cond(Set). However, it is impossible unless some separable assumption.

Proposition 3.9. *There is a "Yoneda" functor*

$$\mathrm{Top}^{T1} \to \mathrm{Cond}(\mathrm{Set})$$

from the category of $T1$ topological spaces to condensed sets. In fact, $\underline{X} \in \mathrm{Cond}_\kappa(\mathrm{Set})$ for some strong limit cardinal κ with $\mathrm{cf}(\kappa) = \lambda > |X|$.

Proof. We need to show that $\underline{X}$ is left Kan extended from the category of κ-small extremally disconnected spaces for some κ. Again by the proof pf Proposition 3.5, it suffices to show for any $S \in \mathrm{CHaus}$ the following natural map

$$\operatorname*{colim}_{S\to T,\ T\in\mathrm{CHaus}_\kappa} C(T, X) \to C(S, X)$$

is an isomorphism. Note that $\mathcal{I}$ be the index category of $S \to T$ with $T \in \mathrm{CHaus}_\kappa$ and $\mathrm{cf}(\kappa) = \lambda$, then $\mathcal{I}$ is λ-filtered. Note also that given such isomorphism, there is no need to sheafify the left hand side since the right hand side is already a sheaf. For injectivity, suppose $S \to T_1 \to X$ and $S \to T_2 \to X$ have the same image in $C(S, X)$, then one can consider $S \to \mathrm{im}(S \to T_1 \times_X T_2) \to X$ which identifies them in the colimit.

As for surjectivity, we have to show for any map $f : S \to X$, it factors as $S \to T \to X$ for some $T \in \mathrm{CHaus}_\kappa$. One can easily check that it suffices to find a map $S \to T$ with $T \in \mathrm{CHaus}_\kappa$ such that $\mathrm{im}(S \times_T S \to X \times X)$ is in the diagonal. In another word, the preimage of (x_1, x_2) is empty for any $x_1 \neq x_2$. Since $\mathcal{I}$ is λ-filtered and $\lambda > |X| = |X \times X|$, it suffices to prove this for for each (x_1, x_2). As $X \times X$ is $T1$, (x_1, x_2) is closed and thus the preimage $T_{(x_1,x_2)}$ is closed, in particular compact Hausdorff. The inverse limit of $T_{(x_1,x_2)}$ with respect to T is empty as $\mathrm{im}(S \times_S S \to X \times X)$ is clearly in the diagonal. Then by compactness, one of them has to be empty, hence we win. ∎

Remark 3.10. It is necessary to require $T1$ (all points are closed) to define a functor from Top to Cond(Set). For example, we can take the Sierpiński space $X = \{s, \eta\}$ where $\{\eta\}$ is open and $\{s\}$ is not. We claim that $\underline{X}$ is not in Cond(Set). Note that $\underline{X}(S)$ is all the open subsets of S. Suppose the opposite that $\underline{X}$ is left Kan extended from the category of κ-small extremally disconnected spaces for some κ. Then by the proof of Proposition 3.5, for any extremally disconnected S, any open subset of S is a uniion of λ-small clopens for some $\lambda < \kappa$[5] which is absurd.

Now we recall a few notions from general topos theory, c.f. [Lur, Lecture 11].

Definition 3.11. Let $\mathcal{X}$ be a topos.

(1) We say $U \in \mathcal{X}$ is quasicompact if every covering has a finite subcover, i.e. for every effective epimorphism $\bigsqcup_{i \in I} U_i \to U$ there is some finite $I_0 \subset I$ such that $\bigsqcup_{i \in I_0} U_i \to U$ is also an effective epimorphism. Moreover, a morphism $U \to V$ is said to be quasicompact, if every morphism $V' \to V$ with V' quasicompact, the fibre product $V' \times_V U$ is quasicompact.
(2) We say U is quasiseparated if for every $V_1 \to U$, $V_2 \to U$ with V_1, V_2 quasicompact, $V_1 \times_U V_2$ is still quasicompact. Moreover, a morphism $U \to V$ is said to be quasiseparated if the diagonal $\Delta : U \to U \times_V U$ is quasicompact.
(3) We say U is qcqs if it is both quasicompact and quasiseparated.
(4) We say $\mathcal{X}$ is coherent is if it has a class of qcqs objects $\mathcal{U}$ generates $\mathcal{X}$, i.e. every object of $\mathcal{X}$ admits a cover from $\mathcal{U}$; and $\mathcal{U}$ is closed under finite products[6].

Remark 3.12. Note that any monomorphism is quasiseparated. In fact, let $\mathcal{C}$ be a essentially small category with finite limits. Then $f : U \to V$ is a monomorphism if and only if the canonical map $U \to U \times_V U$ is an isomorphism. Note that the universal property for $U \times_V U$ means that any parallel arrows $U' \rightrightarrows U$ which are equal after composing with $f : U \to V$ factors through $U \times_V U$. However, U satisfies the universal property for $U \times_V U$ if and only if two parallel arrows $U' \rightrightarrows U$ are equal, i.e. f is a monomorphism.

The following lemma will be used multiple times in sequential without mentioning.

Lemma 3.13. *Let $\mathcal{X}$ be a coherent topos. Then we have the following on quasicompact and quasiseparated objects.*

(1) X is quasicompact (resp. quasiseparated) if and only if $X \to *$ is quasicompact (resp. quasiseparated).
(2) Suppose $X \to Y$ is an effective epimorphism. If X is quasicompact, then so is Y.
(3) Suppose $X \to Y$ is a monomorphism. If Y is quasiseparated, then so is X.

Proof. We first prove (2). Assume $\{U_i \to Y\}_{i \in I}$ is a covering. Then $\{U_i \times_Y X \to X\}_{i \in I}$ is also a covering. Since X is quasicompact, it admits a finite subcover $\{U_i \times_Y X \to X\}_{i \in I_0}$. Since $X \to Y$ is an effective epimorphism, $\{U_i \times_Y X \to Y\}_{i \in I_0}$ is also a covering. Thus $\{U_i \to Y\}_{i \in I_0}$ is a covering.

[5] Here we are using the general fact that if $S \in *_{\kappa-\text{proét}}$ and $U \subset S$ is some open subset, then U is a union of λ-small clopens for some $\lambda < \kappa$.

[6] In particular, it contains a final object of $\mathcal{X}$

We then show (1). The claim on quasicompact is just by definitions. Now assume $X \to *$ is quasiseparated. By the definition of $X \to *$ being quasiseparated, we have $\Delta : X \to X \times X$ being quasicompact. Let Y_1, Y_2 be quasicompact objects with maps $Y_1 \to X \leftarrow Y_2$. Since $*$ is clearly quasiseparated, $Y_1 \times Y_2$ is quasicompact. Therefore, $Y_1 \times_X Y_2 \simeq (Y_1 \times Y_2) \times_{X \times X} X$ is quasicompact as desired. As for the other direction, we need to show that if X is quasiseparated, then $\Delta : X \to X \times X$ being quasicompact, i.e. for any map $Y \to X \times X$ with Y being quasicompact, the fibre product $Y \times_{X \times X} X$ is quasicompact. Since $\mathcal{X}$ is coherent and Y is quasicompact, we can cover Y by finitely many qcqs objects Y_i. Note that since $Y_i \times_{X \times X} X \simeq Y_i \times_Y (Y \times_{X \times X} X)$, they form a covering of $Y \times_{X \times X} X$. Hence by (2), we may reduce to the case where Y is qcqs. Note that since $Y \to X \times X$ factors as $Y \xrightarrow{\Delta} Y \times Y \to X \times X$, we have

$$Y \times_{X \times X} X \simeq Y \times_{Y \times Y} (Y \times_X Y).$$

By assumption $Y \times_X Y$ is quasicompact. Since $\mathcal{X}$ is coherent, $Y \times Y$ is qcqs. Therefore, $Y \times_{X \times X} X$ is quasicompact by the isomorphism above.

As for (3), note that by (1) it suffices to show that $X \to *$ is quasiseparated. However, by the assumption and Remark 3.12, the composition $X \to Y \to *$ is quasiseparated. $\square$

Proposition 3.14. *Let $\mathcal{C}$ be a small category with finite limits and equipped with a finitary Grothendieck topology. Then* $\mathrm{Shv}(\mathcal{C})$ *is a coherent topos. The generators $\mathcal{U}$ can be chosen to be the sheafification of the Yoneda presheaves.*

Proof. A proof can be found in [Lur18, Appendix A.3] or [Lur, Proposition 9, Lecture 11]. $\square$

Example 3.15. Let X be a scheme. Then $\mathrm{Shv}(X_{\mathrm{Zar}})$, $\mathrm{Shv}(X_{\text{ét}})$, $\mathrm{Shv}(X_{\mathrm{fppf}})$, $\mathrm{Shv}(X_{\mathrm{fpqc},\kappa})$ are all coherent since we can just consider affine objects with affine covers, c.f. [Sta, 021E].

Remark 3.16 (Definition). Since Cond(Set) has all limits, colimits (and they are stable under pullbacks), disjoint coproducts and effective quotients, we may define quasicompact and quasiseparated in Cond(Set) as in Definition 3.11.

Restrict the "Yoneda" functor in Proposition 3.9 to the category of compact Hausdorff spaces CHaus, then we can identify the qcqs condensed sets with compact Hausdorff spaces.

Proposition 3.17. *The "Yoneda" functor*

$$\mathrm{CHaus} \to \mathrm{Cond(Set)}$$

induces an equivalence of categories between CHaus *and the full subcategory of qcqs objects in* Cond(Set).

Proof. From Proposition 3.14 we know that CHaus_κ is qcqs in $\mathrm{Cond}_\kappa(\mathrm{Set})$. Since each cover in Cond(Set) can be identified in $\mathrm{Cond}_\kappa(\mathrm{Set})$ for some κ, we see that CHaus is qcqs in $\mathrm{Cond}_\kappa(\mathrm{Set})$. Now let S be a qcqs condensed set. Suppose $S \in \mathrm{Cond}_\kappa(\mathrm{Set})$. Then by Proposition 3.14, we have a cover

$$\bigsqcup_{Y \in \mathrm{CHaus}_\kappa,\ \underline{Y} \to S} \underline{Y} \twoheadrightarrow S.$$

Since S is quasicompact, there is a finite subcover. Notice presentable sheaves are stable under finite coproducts in $\mathrm{Cond}_\kappa(\mathrm{Set})$. Hence there is some $\underline{Y}$ surjects onto S. Since S is quasiseparated,

$\underline{Y} \times_S \underline{Y}$ is quasicompact, then running the above argument again we get a representable sheaf $\underline{X}$ surjects onto $\underline{Y} \times_S \underline{Y}$. Now we can realize S as the coequalizer of $\underline{X} \rightrightarrows \underline{Y}$. Observe that we can identify $\underline{Y} \times_S \underline{Y}$ as the image $\mathrm{im}(\underline{X} \to \underline{Y} \times \underline{Y})$. Hence $\underline{Y} \times_S \underline{Y}$ defines a closed equivalent relation of $\underline{Y} \times \underline{Y} = \underline{Y \times Y}$. Thus $S \simeq \underline{Y}/\underline{R}$ where R a closed equivalent relation. Here we used multiple times that Yoneda embedding $\underline{(-)}$ commutes with limits. □

Remark 3.18. The proof of Proposition 3.17 also shows that Cond(Set) behaves like a coherent topos, i.e. it is generated by the qcqs objects CHaus. In fact the same argument shows that we can take the generating class to be $*_{\text{proét}}$.

Remark 3.19. In the situation of Proposition 3.9, the essential image is the condensed set such that all maps from points are quasicompact. This can be easily checked by encoding the definition. Indeed, for any $x \in \underline{X}(*)$ and any compact Hausdorff S with a map $\underline{S} \to \underline{X}$, we have $\underline{S} \times_{\underline{X}} \underline{\{x\}} \simeq \underline{S \times_X \{x\}}$ where $S \times_X \{x\}$ is closed in S. In general, any quasicompact object can be covered by finitely many qcqs objects and reduce to the argument above. Conversely, for a condensed set S such that all maps from points are quasicompact, we want to show that $S(*)_{\text{top}}$ is $T1$ where $S(*)_{\text{top}}$ just as in Proposition 3.3, is the set $S(*)$ endowed with quotient topology given by $\bigsqcup_{S(T)} T \to S(*)$ for all $T \in *_{\kappa-\text{proét}}$ and all κ. Let $\{x\} \to S$ be quasicompact. By the definition of the topology on $S(*)_{\text{top}}$, it suffices to show that for any compact Hausdorff X with map $X \to S$, the quasicompact object $S' = X \times_S \{x\}$ is closed. However, this is true since one can find a profinite set S'' surjects on to S' such that $S' = \mathrm{im}(S'' \to S)$ where the latter is closed.[7] Then there is a functor $S \to S(*)_{\text{top}}$ to compactly generated spaces, which is the left adjoint of the "Yoneda embedding".

We now describe the discrete condensed objects which play an important role later on.

Lemma 3.20. *Let $\mathcal{C}$ be an essentially small category with finite limits. Then the global sections functor $S \to S(*)$ admits a fully faithful left adjoint*

$$\mathcal{C} \to \mathrm{Cond}(\mathcal{C})$$

given by the constant sheaves valued in C for every $C \in \mathcal{C}$, i.e.

$$C \mapsto (T \mapsto \{\text{locally constant functions on } T \text{ valued in } C\})$$

with $T \in *_{\text{proét}}$. We call the essential image discrete condensed sets. By slightly abusing notations, we will also denote this functor by $C \mapsto \underline{C}$.

Proof. Once we checked the left adjointness, the fully faithfullness will follows immediately since $\underline{C}(*) = C$ for all $C \in \mathcal{C}$. Let $A \in \mathrm{Cond}(\mathcal{C})$. We claim that we have the following identifications

$$\mathrm{Hom}_{\mathrm{Cond}(\mathcal{C})}(\underline{C}, A) \simeq \mathrm{Hom}_{\mathrm{PShCond}(\mathcal{C})}(\underline{\underline{C}}, i(A)) \simeq \mathrm{Hom}_{\mathcal{C}}(C, A(*))$$

where $\underline{\underline{C}}$ is the constant presheaf valued in C and $i(A)$ is viewing A as a presheaf. The first isomorphism is because the sheafification functor is left adjoint to the forgetful functor from presheaves to sheaves. As for the second isomorphism, it suffices to show that any map $\phi : C \to i(A)(*) = A(*)$

[7]The condition that all maps from points are quasicompact is necessary to get a $T1$ space. Note a prior one only has a compactly generated space in the sense of Definition 3.1. Note that the Sierpiński space as in Remark 3.10 is compactly generated but not $T1$.

induces a map from $\underline{\underline{C}}$ to $i(A)$. However, for any $T \in *_{\text{proét}}$, we have $\underline{\underline{C}}(T) = \underline{\underline{C}}(*)$ and a unique map $i(A)(*) \to i(A)(T)$. Hence there is a unique extension defined by composition

$$\underline{\underline{C}}(T) = \underline{\underline{C}}(*) \xrightarrow{\phi} i(A)(*) \to i(A)(T)$$

as desired. □

Example 3.21. Apply $\mathcal{C} = \text{Set}$ in Lemma 3.20, one will get discrete condensed sets. In fact, the functor in Lemma 3.20 is by giving each set discrete topology and using Proposition 3.9, since continuous functions to discrete spaces are exactly locally constant functions to discrete spaces. Moreover, $X \in \text{Cond(Set)}$ is discrete if and only if

$$\operatorname*{colim}_i X(T_i) \to X(T)$$

is a bijection for all $T \in *_{\text{proét}}$. Note that if X is discrete, then the image of $\underline{T} \to X$ is compact hence finite showing it must factor some $\underline{T_i}$. Conversely, suppose $\operatorname{colim}_i X(T_i) \to X(T)$ is a bijection for all $T \in *_{\text{proét}}$. Then we have

$$X(T) \simeq \operatorname*{colim}_i X(T_i) \simeq \operatorname*{colim}_i \underline{X(*)}(T_i) \simeq \underline{X(*)}(T)$$

where the second isomorphism is because T_i's are finite; the third isomorphism is by what we have shown above for discrete $\underline{X(*)}$. Now we win by identifying X and $\underline{X(*)}$.

Example 3.22. Apply $\mathcal{C} = \text{Ab}$ in Lemma 3.20, one will get discrete condensed abelian groups. One interesting example of discrete condensed abelian group would be $\mathbb{Z}[\underline{S}]$ for $S \in \text{Set}$. Indeed, for any $A \in \text{Cond(Ab)}$, we have

$$\text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[\underline{S}], A) \simeq \text{Hom}_{\text{Cond(Set)}}(\underline{S}, A) \simeq \text{Hom}_{\text{Set}}(S, A(*))$$

by Example 3.21. On the other hand,

$$\text{Hom}_{\text{Set}}(S, A(*)) \simeq \text{Hom}_{\text{Ab}}(\mathbb{Z}[S], A(*)) \simeq \text{Hom}_{\text{Cond(Ab)}}(\underline{\mathbb{Z}[S]}, A)$$

by applying Lemma 3.20 to $\mathcal{C} = \text{Ab}$. Hence $\mathbb{Z}[\underline{S}] \simeq \underline{\mathbb{Z}[S]}$ is discrete.

To identify the quasiseparated objects in Cond(Set), we first notice the following observation.

Lemma 3.23. *Let $\mathcal{X}$ be a coherent topos. If $U \in \mathcal{X}$ is quasiseparated then U is a filtered colimit of qcqs objects with injective transition maps.*

Proof. Since $\mathcal{X}$ is coherent, there are some qcqs objects generates U, i.e. $f_I : \bigsqcup_{i \in I} U_i \twoheadrightarrow U$. For each finite subset I' of I, f_I induces a map $f_{I'} : \bigsqcup_{i \in I'} U_i \to U$. Hence we have $U \simeq \operatorname{colim}_{I' \subset I} \operatorname{im}(f_{I'})$. Since each I' is finite, each $\operatorname{im}(f_{I'})$ is quasicompact. On the other hand, since U is quasiseparated and $\operatorname{im}(f_{I'})$ injects into U, $\operatorname{im}(f_{I'})$ is quasiseparated as well. Moreover, the filtered index set is ordered by inclusion, which clearly induces injection between corresponding images. □

Using the lemma above and truncating Cond(Set), we can describe the quasiseparated objects in Cond(Set).

Proposition 3.24. *A condensed set is quasiseparated if and only if it is a filtered colimit of qcqs objects (i.e. Yoneda sheaves given by* CHaus*) with injective transition maps, i.e. closed immersions.*

Proof. As noted above, we have the "if" direction. As for the "only if" direction, assume $U = \operatorname{colim} U_i$ with U_i qcqs and transition maps injective. Then for any maps $f : X \to U$ and $g : Y \to U$ with X and Y quasicompact, we can find some U_i such that f, g factors through U_i. Then $X \times_U Y \simeq X \times_{U_i} Y$ gives the desired conclusion. □

Remark 3.25 (**Warning**)**.** Since qcqs objects in Cond(Set) can be identified with CHaus, one may wonder whether it suffices to take such colimits in Top. However it is false and we must take colimits in Cond(Set). For example, let X be a uncountable compact metric space (e.g. $X = [0,1]$), then the topology on X is totally determined by the countable closed subset of X, i.e. metric spaces are sequential. Hence in Top we have $X \simeq \operatorname{colim}_I f(\mathbb{N} \cup \{\infty\})$ where I is the index category of continuous functions $f : \mathbb{N} \cup \{\infty\} \to X$. However, this is not the colimit in Cond(Set). Indeed, if it were, then $\operatorname{colim}_I f(\mathbb{N} \cup \{\infty\})$ will be a qcqs object. Hence the identity map will factor through $f(\mathbb{N} \cup \{\infty\})$ for some f which is absurd. But if I is countable, then some filtered system $\{\mathbb{N} \cup \{\infty\}\}_I$ will have the same colimit in Top and Cond(Set) by the Baire category theorem. Nevertheless, it is not true in general that a countable filtered colimit of compact Hausdorff spaces with injective transition maps. For example, the colimit of $\{[1/i, 1] \cup \bigcup_{j>i} \{1/j\}\}_{i \in \mathbb{N}}$ is $[0,1]$ in Top but something else in Cond(Set) by the same reasoning above.

Remark 3.26. Recall that a topological space X is called weakly Hausdorff means that for any $S \in$ CHaus maps into X, the image $\operatorname{im}(S \to X) \in$ CHaus. Then the above discussion shows that compactly generated weakly Hausdorff spaces (CGWH) embedds into quasiseparated objects of condensed sets. In particular, condensed sets contain locally compact Hausdorff spaces. Summarizing the situation, we have the following diagram

$$\begin{array}{llcccccccc}
\text{Cond(Set)}: & \text{quasicompact projective} & \hookrightarrow & *_{\text{proét}} & \hookrightarrow & \text{qcqs} & \hookrightarrow & \text{quasiseparated} & \hookleftarrow & \text{discrete} \\
 & \uparrow \simeq & & \uparrow = & & \uparrow \simeq & & \uparrow & & \uparrow \simeq \\
\text{Top}: & \text{extremally disconnected} & \hookrightarrow & *_{\text{proét}} & \hookrightarrow & \text{CHaus} & \hookrightarrow & \text{CGWH} & \hookleftarrow & \text{Set}
\end{array}$$

Notation 3.27. From now on, when a topological space X is CGWH, we no longer distinguish its associated sheaf $\underline{X}$ and itself. By slightly abuse of notations, we will all denote as X.

We record a few useful lemmas which characterize maps between quasicompact, quasiseparated, qcqs objects from [Cas23].

Lemma 3.28. *Let $S \to T$ be a map in* Cond(Set) *with S quasicompact and T qcqs. Then we have the following:*

(1) $S \to T$ is surjective if and only if $S(*) \to T(*)$ is surjective.

(2) Assume further that S is qcqs. Then $S \to T$ is isomorphic if and only if $S(*) \to T(*)$ is isomorphic.

Proof. For (1), the "if" direction is straightforward since $*$ is extremally disconnected, i.e. projective. On the other hand, as S is quasicompact, there is a qcqs object S' surjects onto S, giving $S'(*) \twoheadrightarrow S(*)$. Hence we have $S'(*) \twoheadrightarrow S(*) \twoheadrightarrow T(*)$ where the composition is a surjection in CHaus. Therefore, $S' \to T$ is surjective and consequently, $S \to T$ is surjective.

The proof of (2) relies on (1) and the following lemma. □

Lemma 3.29. *Let $S \to T$ be a map in* $\mathrm{Cond(Set)}$ *with S qcqs and T quasiseparated. Then the following are equivalent:*

(1) $S \to T$ is injective;

(2) $S(*) \to T(*)$ is injective;

(3) For any $T' \to T$ in $\mathrm{Cond(Set)}$ with $\mathrm{im}(T'(*) \to T(*)) \subset \mathrm{im}(S(*) \to T(*))$, the map $T' \to T$ can be uniquely lift to $T' \to S$ along $S \to T$.

Proof. (1) $\Leftrightarrow$ (2): Clearly we have (1) $\Rightarrow$ (2). As for (1) $\Leftarrow$ (2), note that in general (1) is equivalent to $S \to S \times_T S$ is an isomorphism by Remark 3.12. However, since $\mathrm{im}(S \to T)$ is qcqs, we have $S \times_T S \simeq S \times_{\mathrm{im}(S \to T)} S$ is qcqs. Thus we may reduce everything to CHaus which is clear that injectivity is determined by their underlying sets.

(1) $\Leftrightarrow$ (3): For (1) $\Rightarrow$ (3), we may replace T' by $\mathrm{im}(T' \to T)$ then T' is quasiseparated. Thus $T' \simeq \mathrm{colim}_I T'_i$ is a filtered colimit of qcqs objects with injective transition maps by Proposition 3.24. By checking in CHaus we know that for each i, T'_i factors through S uniquely. Therefore, the colimit also factor through S uniquely. Clearly we have (2) $\Leftarrow$ (3). □

4. Condensed Cohomology

We have already seen in Theorem 2.17 that Cond(Ab) has enough projective objects. And general nonsense about topos in abelian groups implies it has enough injective objects. Hence we can consider the derived category $D(\mathrm{Cond}(\mathrm{Ab}))$. For now, we just work with triangulated derived category. Later we will work with the derived ∞-category which we will denote as $\mathcal{D}(\mathrm{Cond}(\mathrm{Ab}))$. The following Proposition follows from the general facts about derived category.

Proposition 4.1. *In $D(\mathrm{Cond}(\mathrm{Ab}))$ we have the following:*

(1) $D(\mathrm{Cond}(\mathrm{Ab}))$ is generated by compact and projective objects. One can take the generators (under shifts and colimits) by $\mathbb{Z}[S]$ where S is extremally disconnected.

(2) Derived tensor and derived Hom exist by Remark 2.22 and Cond(Ab) has enough injectives, respectively.

(3) Derived internal Hom exists. By Remark 2.23, $R\underline{\mathrm{Hom}}(M,N)(S) = R\mathrm{Hom}(M\otimes\mathbb{Z}[S],N)$ for S extremally disconnected.

(4) For any $P,M,N\in D(\mathrm{Cond}(\mathrm{Ab}))$, we have $\mathrm{Hom}(P\otimes^L M,N)=\mathrm{Hom}(P,R\underline{\mathrm{Hom}}(M,N)$.

(5) For any $P,M,N\in D(\mathrm{Cond}(\mathrm{Ab}))$, we have $R\mathrm{Hom}(P\otimes^L M,N)=R\mathrm{Hom}(P,R\underline{\mathrm{Hom}}(M,N)$ by (3) and (4).

Now we turn to the cohomology of condensed abelian groups. We will focus on the cases with constant coefficient $\mathbb{Z}$ and $\mathbb{R}$.

Definition 4.2. For any $S\in\mathrm{Cond}(\mathrm{Set})$ and $F\in\mathrm{Cond}(\mathrm{Ab})$, we define $R\Gamma_{\mathrm{Cond}}(S,F) := R\mathrm{Hom}(\mathbb{Z}[S],F)$.

Remark 4.3. If $X\in\mathrm{CHaus}$, then by adjunction between Cond(Set) and Cond(Ab) and Yoneda lemma (c.f. the proof of Theorem 2.17), we have

$$\Gamma_{\mathrm{Cond}}(X,F) := F(X)\simeq \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[X],F).$$

This partially justifies the above definition is "taking the cohomology" for compact Hausdorff spaces. However, this does not show that it is the same as the usual sheaf cohomology when F is constant (since we are deriving different functors). But we will see later that this is the same as the sheaf cohomology.

Remark 4.4. If $X\in\mathrm{CHaus}$, then the open topology and closed topology on X gives the same topos. In fact, because X is normal, for any open neighborhood U of some point x, one can find $x\in V\subset Z\subset U$ for some closed Z and open V.

Remark 4.5. We now give a projective resolution of $\mathbb{Z}[X]$ for $X\in\mathrm{CHaus}$ using hypercovers[8]. We follow the notations and treatments as in [Con03]. By Remark 2.16, we can find a extremally disconnected space S_0 surjects onto X. By induction, assume we have a n-truncated object $S_{\le n}$ of extremally disconnected spaces. Then one may take a surjection from a extremally disconnected space S_{n+1} onto $(\mathrm{cosk}_n(S_{\le n}))_{n+1}$ which we denote as ϕ. In this way we get a $(n+1)$-truncated object $S_{\le n+1}$ of extremally disconnected spaces by composing the degeneracy maps of $\mathrm{cosk}_n(S_{\le n})$ with the section of ϕ, and composing the face maps of $\mathrm{cosk}_n(S_{\le n})$ with ϕ. Thus, we get a hypercover

[8] By hypercovers, we always mean proper surjective hypercovers in suitable sites.

$S_\bullet$ of X. It is easy to verify that this is also a universal hypercover, i.e. retains hypercover under base change. Let $P^\bullet = \mathbb{Z}[S_\bullet]$. Then by some general nonsense (c.f. [Sta, 01GF]) we get a projective resolution $P^\bullet \to \mathbb{Z}[X]$.

Theorem 4.6. *There are natural functorial isomorphisms for $X \in \mathrm{CHaus}$:*

$$R\Gamma_{\mathrm{Cond}}(X, \mathbb{Z}) \simeq R\Gamma_{\mathrm{Sheaf}}(X, \mathbb{Z})$$

where $\mathbb{Z}$ is endowed with the discrete topology.

The proof strategy is the following: we first show that $F(U) := R\Gamma_{\mathrm{Cond}}(U, \mathbb{Z})$ satisfies topological descent, i.e. $F(X) \simeq R\Gamma_{\mathrm{Sheaf}}(X, F^\sharp)$ where $F^\sharp$ is the sheafification of F; we then show that $F^\sharp$ is isomorphic to the constant sheaf $\mathbb{Z}$ by checking stalks. We first observe the following.

Lemma 4.7. *For any $S \in *_{\text{proét}}$, we have*

$$R\Gamma_{\mathrm{Cond}}(S, M) = \Gamma_{\mathrm{Cond}}(S, M)$$

for any $M \in \mathrm{Cond}(\mathrm{Ab})$ discrete.

Proof. To prove higher cohomologies vanish, by [Sta, 01EV] it suffices to show that for any $S' \twoheadrightarrow S$, the Čech complex

$$\Gamma_{\mathrm{Cond}}(S, M) \to \Gamma_{\mathrm{Cond}}(S', M) \to \Gamma_{\mathrm{Cond}}(S' \times_S S', M) \to \cdots$$

is acyclic. Note that one can take $S' \twoheadrightarrow S$ to be a cofiltered limit of $S_i' \twoheadrightarrow S_i$ where each S_i' and S_i are finite[9]. However, note that $M(S) \simeq \operatorname{colim} M(S_i)$ by the fact that M is discrete, c.f. Example 3.21. Then the above assertion reduces to the case for S and S' are both finite. This case is clear since $S' \twoheadrightarrow S$ splits. □

Remark 4.8. In the case where $M = \mathbb{Z}$, we can also identify $\Gamma_{\mathrm{Cond}}(S, M)$ with $C(S, \mathbb{Z})$ by adjunction and the fact that both S and $\mathbb{Z}$ are compactly generated.

Now we prove Theorem 4.6.

Proof of Theorem 4.6. We start by constructing a natural morphism of topoi α with

$$\alpha = (\alpha^*, \alpha_*) : \mathrm{Cond}_{/X}(\mathrm{Ab}) \to \mathrm{Shv}(X, \mathrm{Ab})$$

where

$$\alpha^*(F) = ((\pi : Y \to X) \mapsto \Gamma_{\mathrm{Sheaf}}(Y, \pi^* F))^\sharp$$
$$\alpha_*(F) = ((U \subset X) \mapsto \Gamma_{\mathrm{Cond}}(U, F)).^{10}$$

We first verify that α_* is well-defined, i.e. for any open cover $\bigcup_i U_i = U$ we have

$$\Gamma_{\mathrm{Cond}}(U, F) \xrightarrow{\simeq} \mathrm{Tot}(\prod \Gamma_{\mathrm{Cond}}(U_i, F) \to \prod \Gamma_{\mathrm{Cond}}(U_i \times_X U_j, F) \to \cdots).$$

Since U is locally compact Hausdorff, it is a colimit of compact Hausdorff spaces. Hence it suffices to show the case where U itself is compact Hausdorff. Without loss of generality, we may assume $U = X$. Now since X is compact, every open cover can be refined by a finite cover. Hence by

[9] For example, let $S = \lim S_i$, then one can take $S_i' = S' \times_{S_i} S$ and take a profinite representation of S_i'.

[10] Open subsets of locally compact Hausdorff spaces are locally compact Hausdorff spaces hence again condensed sets.

induction we may assume that $U \cup V = X$. Since $\Gamma_{\text{Cond}}(-, F) = \text{Hom}_{\text{Cond(Ab)}}(\mathbb{Z}[-], F)$ takes colimits to limits, it suffices to show we have the following short exact sequence:

$$0 \to \mathbb{Z}[U \cap V] \to \mathbb{Z}[U] \oplus \mathbb{Z}[V] \to \mathbb{Z}[X] \to 0.$$

The only nontrivial part is checking surjectivity. Since $X \in \text{CHaus}$, it is normal. Hence one can find $U' \cup V' = X$ with U', V' compact and $U' \subset U$, $V' \subset V$. Using this fact, it is easy to check $\mathbb{Z}[U] \oplus \mathbb{Z}[V] \to \mathbb{Z}[X]$ is surjective.

Now note that we have a natural identification $\Gamma_{\text{Cond}}(X, F) = \Gamma_{\text{Sheaf}}(X, \alpha_*(F))$. On the other hand, α_* is left exact and α^* is exact with $\alpha^*(\mathbb{Z}) = \mathbb{Z}$. Passing to the derived category we have a natural isomorphism

$$R\Gamma_{\text{Sheaf}}(X, R\alpha_*\alpha^*(\mathbb{Z})) \simeq R\Gamma_{\text{Cond}}(X, \mathbb{Z})$$

Therefore, it suffices to show that $R\alpha_*(\mathbb{Z}) \simeq \mathbb{Z}$. We will check this on stalks, so fix $x \in X$. Since $X \in \text{CHaus}$, in particular locally compact Hausdorff, the closed neighborhoods V of x is cofinal in the neighborhoods U of x. Hence we have

$$(R\alpha_*(\mathbb{Z}))_x = \operatorname*{colim}_{x \in U} R\Gamma_{\text{Cond}}(U, \mathbb{Z}) \simeq \operatorname*{colim}_{x \in V} R\Gamma_{\text{Cond}}(V, \mathbb{Z})$$

Then we can take a hypercover $S_\bullet \to X$ as in Remark 4.5 giving a hypercover $S_\bullet \times_X V$ of V. Note that for each $S_i \times_X V$, it is still a profinite set (even though not extremally disconnected anymore). Hence $\mathbb{Z}[S_\bullet] \otimes \mathbb{Z}[V]$ is still a resolution of $\mathbb{Z}[V]$. By Lemma 4.7 and a spectral sequence argument, $R\Gamma_{\text{Cond}}(V, \mathbb{Z})$ can be computed by

$$\Gamma_{\text{Cond}}(S_0 \times_X V, \mathbb{Z}) \to \Gamma_{\text{Cond}}(S_1 \times_X V, \mathbb{Z}) \to \cdots$$

Passing to the colimits of $x \in V$, we have the following

$$\begin{aligned}(R\alpha_*(\mathbb{Z}))_x &\simeq \operatorname*{colim}_{x \in V}(\Gamma_{\text{Cond}}(S_0 \times_X V, \mathbb{Z}) \to \Gamma_{\text{Cond}}(S_1 \times_X V, \mathbb{Z}) \to \cdots) \\ &\simeq \Gamma_{\text{Cond}}(S_0 \times_X \{x\}, \mathbb{Z}) \to \Gamma_{\text{Cond}}(S_1 \times_X \{x\}, \mathbb{Z}) \to \cdots \\ &\simeq \Gamma_{\text{Cond}}(\{x\}, \mathbb{Z}) \\ &\simeq \mathbb{Z}.\end{aligned}$$

□

Remark 4.9. The above proof works for arbitrary discrete abelian groups, i.e. for any discrete abelian group M one has

$$R\Gamma_{\text{Cond}}(X, M) \simeq R\Gamma_{\text{Sheaf}}(X, M).$$

One can give a slightly different proof using Remark 4.5 and hyperdescent. We start by making a similar observation as Lemma 4.7.

Lemma 4.10. *For any $S \in *_{\text{proét}}$, we have*

$$R\Gamma_{\text{Sheaf}}(S, F) = \Gamma_{\text{Sheaf}}(S, F)$$

for any abelian sheaf F.

Proof. Observe that any open cover of S can be refined by a finite disjoint clopen cover, c.f. [Sta, 08ZZ]. This shows that the global sections functor is exact, hence the assertion. □

Remark 4.11. Similarly as the condensed cohomology case, if $F = \mathbb{Z}$ then we have $\Gamma_{\text{Sheaf}}(S, \mathbb{Z}) \simeq C(S, \mathbb{Z})$ by observing that all locally constant maps to a discrete space is continuous.

An Alternate Proof of Theorem 4.6. Note that once taken a hypercover of extremally disconnected spaces $S_\bullet \to X$, by the fact that $\mathbb{Z}[S_\bullet]$ is a projective resolution, we can compute $R\Gamma_{\text{Cond}}(X, \mathbb{Z})$ by

$$\Gamma_{\text{Cond}}(S_0, \mathbb{Z}) \to \Gamma_{\text{Cond}}(S_1, \mathbb{Z}) \to \cdots$$

Then we win by the hyperdescent of sheaf cohomology, c.f. [Sta, 0D91,09XA] and Lemma 4.10. □

Now we turn to the condensed cohomology with continuous coefficient.

Proposition 4.12. *For any $X \in$ CHaus and any hypercover of profinite sets $S_\bullet \twoheadrightarrow X$, the complex of Banach spaces (where $\mathbb{R}$ is endowed with the usual Euclidean topology and the Banach spaces are endowed with supreme norm)*

$$0 \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

has the property that if $f \in C(S_i, \mathbb{R})$ for $i \geq 1$ with $df = 0$, then for any $\epsilon > 0$ there exists a $g \in C(S_{i-1}, \mathbb{R})$ with $||g|| \leq (1 + \epsilon)||f||$ and $||f - dg|| \leq \epsilon||f||$.

Proof. We first assume that X is finite. Then $S_\bullet \to X$ splits and thus is a homotopy equivalence by [Lur06, Lemma 6.1.3.16]. Note that all maps from finite sets to $\mathbb{R}$ is continuous. In another word, we have a continuous section $h_i : S_i \to S_{i+1}$. Then for any $f \in C(S_i, \mathbb{R})$ for $i \geq 1$ and $df = 0$, we can take $g = f \circ h_i$ which gives $dg = f$ and $||g|| \leq ||f||$.

Now we fix an arbitrary $X \in$ CHaus and an arbitrary hypercover of profinite sets $S_\bullet \to X$. Let $f \in C(S_i, \mathbb{R})$ for $i \geq 1$ with $df = 0$ and $\epsilon > 0$. Then for any $x \in X$, the previous case implies that there is a continuous function g_x on $S_{i-1} \times_X \{x\}$ such that $dg_x = f_x := f|_{S_i \times_X \{x\}}$ and $||g_x|| \leq ||f_x||$. By Tietze's extension theorem, there is a $\tilde{g}_x$ extends to S_{i-1} and $||\tilde{g}_x|| = ||g_x||$. Now we let $W_x = \{s \in S_i \mid |(f - d\tilde{g}_x)(s)| \geq \epsilon||f||\}$. It is closed by the continuity of $f - d\tilde{g}_x$. Now since $\pi_i : S_i \to X$ is proper, $\pi_i(W_x)$ is compact and $x \notin \pi_i(W_x)$, we take $U_x = X \backslash \pi_i(W_x)$ to a open neighborhood of x. By construction,

$$||(f - d\tilde{g}_x)|_{S_i \times_X U_x}|| < \epsilon||f||.$$

Now since X is compact, one can cover X finitely many U_{x_i}'s. Then use partition of unit to get $\sum \rho_i = 1$ with $\text{supp}(\rho_i) \subset U_{x_i}$. Then we win by taking $g = \sum \rho_i \tilde{g}_{x_i}$. □

Proposition 4.12 has the following theorem as a consequence.

Theorem 4.13. *For any $X \in$ CHaus, we have*

$$R\Gamma_{\text{Cond}}(X, \mathbb{R}) = \Gamma_{\text{Cond}}(X, \mathbb{R}) \simeq C(X, \mathbb{R}).$$

Moreover, for any hypercover of profinite sets $S_\bullet \twoheadrightarrow X$, the complex of Banach spaces (where $\mathbb{R}$ is endowed with the usual Euclidean topology and the Banach spaces are endowed with supreme norm)

$$0 \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

has the property that if $f \in C(S_i, \mathbb{R})$ for $i \geq 1$ with $df = 0$, then for any $\delta > 0$ there exists a $g \in C(S_{i-1}, \mathbb{R})$ with $dg = f$ and $||g|| \leq (1 + \delta)||f||$.

Proof. As in Remark 4.5, we can take a hypercover of X by profinite sets $S_\bullet$. Hence it suffices to show that

$$0 \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

is exact in nonzero degrees. Fix $f \in C(S_i, \mathbb{R})$ for $i \geq 1$ with $df = 0$ and $\delta > 0$. Choose $0 < \epsilon < 1$ such that $(1+\epsilon)/(1-\epsilon) < 1 + \delta$. Now we proceed the following inductive steps:

(1) Set $f^{(0)} = f$, then we get $g^{(0)}$ with $||g^{(0)}|| \leq (1+\epsilon)||f||$ and $||f^{(0)} - dg^{(0)}|| \leq \epsilon||f||$.
(2) Set $f^{(1)} = f^{(0)} - dg^{(0)}$, then we get $g^{(1)}$ with $||g^{(1)}|| \leq (1+\epsilon)||f^{(1)}|| \leq \epsilon(1+\epsilon)||f||$ and $||f^{(1)} - dg^{(1)}|| \leq \epsilon||f^{(1)}|| \leq \epsilon^2||f||$.
(3) Iterate this process.

Then we take $g = \lim_{n\to\infty} \sum_{i=0}^n g^{(i)}$. The limit exists because we are working in Banach spaces. Now we have

$$||g|| \leq (1+\epsilon)(\sum_{i=0}^{\infty} \epsilon^i)||f|| \leq (1+\epsilon)/(1-\epsilon)||f|| \leq (1+\delta)||f||.$$

with $||f - dg|| = 0$ as desired. □

Remark 4.14. We remark here that the condensed cohomology with $\mathbb{R}$ coefficient is the "correct" thing. Recall that a classical sheaf on a topological space $F \in \mathrm{Shv}(X)$ is called soft if $F(X) \to F(Z)$ is surjective when $Z \subset X$ is compact. Any flasque sheaf is soft. And for any locally compact Hausdorff space X, the sheaf of continuous functions $C(-, \mathbb{R})$ with value in $\mathbb{R}$ is soft. In fact, for any $s \in C(Z, \mathbb{R})$ it is represented by a continuous function f defined on a open neighborhood of Z (by the identification $\Gamma(Z, C) \simeq \mathrm{colim}_{Z\subset U} \Gamma(U, C)$). Then we are done by Tietze's extension theorem. Similarly, for smooth manifolds, the sheaf $C^\infty(-, \mathbb{R})$ of smooth functions is soft. Soft sheaves are useful since they are acyclic objects for compactly supported cohomology, i.e.

$$R\Gamma_c(X, F) = \Gamma_c(X, F)$$

for F soft. As an immediate application, one can prove de Rham comparison using Poincaré lemma.

5. Locally Compact Abelian Groups

We want to consider a nice class of topological abelian groups.

Definition 5.1. A topological abelian group A is called locally compact if the underlying topological space is locally compact, i.e. for any $x \in A$, it has basis of compact Hausdorff neighborhoods. We will denote the category of locally compact abelian groups with continuous morphisms as LCA.

Example 5.2. It is easy to see that the following examples are locally compact abelian groups: integers $\mathbb{Z}$ (with discrete topology), real numbers $\mathbb{R}$ (with Euclidean topology), p-adic numbers $\mathbb{Q}_p$, torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

Remark 5.3. The category LCA is not abelian. For example one can take $\mathbb{R}^{\mathrm{disc}} \to \mathbb{R}^{\mathrm{eucl}}$. Even though it is not abelian, it is quasi-abelian.

Definition 5.4. An additive category $\mathcal{C}$ is called quasi-abelian if it satisfies the following conditions:

(1) Kernels exist and are stable under pushouts.
(2) Cokernels exist and are stable under pullbacks.

And we call a map $f : A \to B$ strict if $A/\ker(f) \to \overline{\mathrm{im}(f)}$ is an isomorphism. A complex is called strictly exact if it is exact and each differentials are strict.

Remark 5.5. In the case of LCA, for a morphism $f : A \to B$, kernel $\ker(f)$ is defined to be the usual one since kernel is closed. Cokernel of f are defined to be $B/\overline{\mathrm{im}(f)}$. Then one can see that a morphism is strict if and only if $A/\ker(f) \to B$ is a closed embedding. In particular, a short exact sequence

$$0 \to A \to B \to C \to 0$$

is strict if and only if $A \to B$ is a closed embedding and $B \to C$ is open (or equivalently topological quotient).

Definition 5.6. For $A, B \in \mathrm{LCA}$, we endow $\mathrm{Hom}_{\mathrm{LCA}}(A, B)$ with compact open topology, i.e. the topology generated by the subsets $U(K, V) := \{f \in \mathrm{Hom}_{\mathrm{LCA}}(A, B) \mid f(K) \subset V\}$ with $K \subset A$ compact and $V \subset B$ open. We denote this topological abelian group by $\underline{\mathrm{Hom}}_{\mathrm{LCA}}(A, B)$. This gives an "internal Hom" functor in the category of topological abelian groups, i.e. for $A, B, C \in \mathrm{LCA}$

$$\mathrm{Hom}_{\mathrm{TopAb}}(A \times B, C) = \mathrm{Hom}_{\mathrm{TopAb}}(A, \underline{\mathrm{Hom}}_{\mathrm{LCA}}(B, C))$$

Remark 5.7 (**Warning**)**.** It is easy to see that $\underline{\mathrm{Hom}}_{\mathrm{LCA}}(A, B)$ is still Hausdorff but in general not locally compact. For example, consider $\underline{\mathrm{Hom}}_{\mathrm{LCA}}(\bigoplus_I \mathbb{Z}, \mathbb{R}) \simeq \prod_I \mathbb{R}$, which is not locally compact when I is infinite. Hence $\underline{\mathrm{Hom}}_{\mathrm{LCA}}$ is not an internal Hom functor in the category LCA.

The following classical results on LCA can be found in [DE14].

Theorem 5.8. *Let $\mathbb{D}(-) := \underline{\mathrm{Hom}}_{\mathrm{LCA}}(-, \mathbb{T})$ denoting the Pontryagin duality functor. It has the following properties:*

(1) $\mathbb{D}$ induces a contravariant autoequivalence on LCA *which interchanges compact abelian groups and discrete abelian groups.*
(2) $\mathbb{D}$ is idempotent, i.e. $\mathrm{Id} \simeq \mathbb{D} \circ \mathbb{D}$.

(3) $\mathbb{D}$ preserves strictly exact sequences.

Theorem 5.9. *Let $A \in \mathrm{LCA}$. Then $A \simeq \mathbb{R}^n \times A'$ with A' being an extension of a discrete abelian group and a compact abelian group.*

We then connect the theory of locally compact abelian groups with the theory of condensed abelian groups. We need the Breen-Deligne resolution for (condensed) abelian groups. We just state the result here and give a proof in Section 5.1.

Proposition 5.10. *Let $A \in \mathrm{Cond(Ab)}$. Then there is a functorial resolution of A given by*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A \times A] \xrightarrow{m} \mathbb{Z}[A] \xrightarrow{\epsilon} A \to 0.$$

where $m([a_1, a_2]) = [a_1 + a_2] - [a_1] - [a_2]$ and $\epsilon([a]) = a$.

Proposition 5.11. *Let $A, B \in \mathrm{LCA}$. Then*

$$\underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(A, B) \simeq \underline{\underline{\mathrm{Hom}_{\mathrm{LCA}}(A, B)}}.$$

Note that thanks to $\underline{\mathrm{Hom}}_{\mathrm{LCA}}(A, B)$ being Hausdorff, by Proposition 3.9, $\underline{\underline{\mathrm{Hom}_{\mathrm{LCA}}(A, B)}}$ is well-defined.

Proof. We start by constructing such a map. Note that for any S extremally disconnected, we construct the following map

$$\phi : \underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(A, B)(S) = \mathrm{Hom}_{\mathrm{Cond(Ab)}}(A \otimes \mathbb{Z}[S], B) \to C(S, \underline{\mathrm{Hom}}_{\mathrm{LCA}}(A, B))$$

given by $\phi(f)(s) = (a \mapsto f(*)(a \otimes s))$ where $f(*)$ is f evaluating at a point. We now show that ϕ is well-defined, i.e. $\phi(f)$ is continuous. By Corollary 5.10, we have a resolution of A as condensed abelian groups

$$\mathbb{Z}[A \times A] \xrightarrow{m} \mathbb{Z}[A] \xrightarrow{\epsilon} A \to 0.$$

Since S is extremally disconnected, $\mathbb{Z}[S]$ is projective, hence tensoring the above resolution gives a resolution of $A \otimes \mathbb{Z}[S]$ (by slightly abusing notations)

$$\mathbb{Z}[A \times A \times S] \xrightarrow{m} \mathbb{Z}[A \times S] \xrightarrow{\epsilon} A \otimes \mathbb{Z}[S] \to 0.$$

Now since $A \otimes \mathbb{Z}[S]$ is quotient of $\mathbb{Z}[A \times S]$, any map $A \otimes \mathbb{Z}[S] \to B$ is determined by a map $\mathbb{Z}[A \times S] \to B$, where the latter is determined by a map of condensed sets $A \times S \to B$, i.e. a continuous map $A \times S \to B$ on the underlying topological spaces. Hence we win by the $\underline{\mathrm{Hom}}_{\mathrm{LCA}}$ being an "internal Hom".

To show this map is an isomorphism, note that we have the following identifications by applying $\mathrm{Hom}_{\mathrm{Cond(Ab)}}(-, B)$ to the resolution above.

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(A, B)(S) &= \mathrm{Hom}_{\mathrm{Cond(Ab)}}(A \otimes \mathbb{Z}[S], B) \\
&\simeq \{f \in \mathrm{Hom}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[A \times S], B) \mid f \circ m = 0\} \\
&\simeq \{f \in \mathrm{Hom}_{\mathrm{Cond(Set)}}(A \times S, B) \mid f \circ m = 0\} \\
&\simeq \{f \in C(A \times S, B) \mid f(a_1 + a_2, s) = f(a_1, s) + f(a_2, s)\} \\
&\simeq C(S, \underline{\mathrm{Hom}}_{\mathrm{LCA}}(A, B)).
\end{aligned}$$

□

We also want to compare the derived category of LCA (constructed in [HS07]) and the derived category of Cond(Ab). We start by making the following observation.

Lemma 5.12. *The Yoneda functor* LCA $\to$ Cond(Ab) *sends stricly exact sequences to exact sequences.*

Proof. Clearly left exactness is preserved. Hence it suffices to show that for any strictly surjective map $f : A \twoheadrightarrow B$ in LCA gives surjective map in Cond(Ab). Unwinding the definition, it suffices to show that for any continuous map $g : S \to B$ for some extremally disconnected S, there is a lift of g to a compact subset Z of A (here we used $g(S)$ is compact, otherwise one need to find a compact cover of $g(S)$), i.e. the composition $S \to Z \hookrightarrow A \twoheadrightarrow B$ is g. Since S is projective, it suffices to find a compact subset Z of A such that $g(S) \subset f(Z)$. Since f is strict, it is open by Remark 5.5. Hence compact neighborhood K of the identity in A maps to a compact neighborhood $f(K)$ of the identity in B. Then we win by covering $g(S)$ with finitely many $f(K + a)$'s. □

Definition 5.13. Let $\mathrm{Ch}^b(\mathrm{LCA})$ be category of the bounded chain complexes of LCA and $D^b(\mathrm{LCA})$ be $\mathrm{Ch}^b(\mathrm{LCA})$ inverting maps whose cones are strictly exact. We call $D^b(\mathrm{LCA})$ the derived category of LCA.

Corollary 5.14. *The Yoneda functor induces a functor of derived categories* $D^b(\mathrm{LCA}) \to D(\mathrm{Cond(Ab)})$.

Remark 5.15. The definition of $R\mathrm{Hom}$ in $D^b(\mathrm{LCA})$ is a little involved. Hence we just record the following facts in [HS07] that will be used later.

(1) $R\mathrm{Hom}_{\mathrm{LCA}}(\prod_I \mathbb{T}, M) \simeq \bigoplus_I M[-1]$ for any M discrete.
(2) $R\mathrm{Hom}_{\mathrm{LCA}}(\prod_I \mathbb{T}, \mathbb{R}) \simeq 0$.

Theorem 5.16. *The functor* $D^b(\mathrm{LCA}) \to D(\mathrm{Cond(Ab)})$ *is fully faithful.*

To prove the theorem we need to compute $R\mathrm{Hom}$ in $D(\mathrm{Cond(Ab)})$ between locally compact abelian groups. By Theorem 5.9, one only needs to compute the following.

Proposition 5.17. *We have the following in* $D(\mathrm{Cond(Ab)})$.

(1) $R\underline{\mathrm{Hom}}(\prod_I \mathbb{T}, M) \simeq \bigoplus_I M[-1]$ *for any* M *discrete.*
(2) $R\underline{\mathrm{Hom}}(\prod_I \mathbb{T}, \mathbb{R}) \simeq 0$.

We need the following formal consequence of Proposition 5.10.

Lemma 5.18. *For any* $A, M \in \mathrm{Cond(Ab)}$ *and* S *extremally disconnected, we have*

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q(A^{r_{p,j}} \times S, M) \Rightarrow R^{p+q}\underline{\mathrm{Hom}}(A, M)(S).$$

Proof. Apply $R\underline{\mathrm{Hom}}(-, M)$ to the resolution in Proposition 5.10. □

Proof of Proposition 5.17. As for (1), we first show the case where I is finite. In this case, by decomposing into direct sum, we may reduce to the case where $I = \{*\}$. By the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

to show $M[-1] \simeq R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \to R\underline{\mathrm{Hom}}(\mathbb{T}, M)$ is an isomorphism, it suffices to show that $R\underline{\mathrm{Hom}}(\mathbb{R}, M) \simeq 0$. We must show that applying $R\underline{\mathrm{Hom}}(-, M)$ on $0 \to \mathbb{R}$ induces an isomorphism. By Lemma 5.18, it suffices to show that

$$H^q(\mathbb{R}^r \times S, M) \to H^q(S, M)$$

induced by $S \to \mathbb{R}^r \times S$ is an isomorphism for all $r \in \mathbb{N}$ and S extremally disconnected. Note that $\mathbb{R} \times S \simeq \mathrm{colim}_m([-m, m] \times S)$ in Cond(Set). Since the interior $(-m, m)$ of $[-m, m]$ cover $\mathbb{R}$ and therefore any map from a compact Hausdorff space to $\mathbb{R}$ factor through some $[-m, m]$[11]. Hence we have $R\Gamma(\mathbb{R}^r \times S, M) \simeq R\lim_m R\Gamma([-m, m]^r \times S, M)$. Using Theorem 4.6 and Remark 4.9, we can identify the condensed cohomology here with the sheaf cohomology. Then we win by the homotopy invariance of the sheaf cohomology.

For general I, it suffices to show that

$$\operatorname*{colim}_{\text{finite } J \subset I} R\underline{\mathrm{Hom}}(\prod_J \mathbb{T}, M) \simeq R\underline{\mathrm{Hom}}(\prod_I \mathbb{T}, M).$$

Again, we use Lemma 5.18 and Theorem 4.6 (and Remark 4.9) to reduce to the sheaf cohomology case, which is clear.

Now we prove (2). To simplify the notation, we let $A = \prod_I \mathbb{T}$. By Lemma 5.18 and Theorem 4.13, $R\underline{\mathrm{Hom}}(A, \mathbb{R})(S)$ is computed by

$$0 \to C(A \times S, \mathbb{R}) \to C(A^2 \times S, \mathbb{R}) \to \cdots \to \bigoplus_{j=1}^{n_p} C(A^{r_{p,j}} \times S, \mathbb{R}) \to \cdots$$

Note that multiplication by 2 on $R\underline{\mathrm{Hom}}(A, \mathbb{R})(S)$ can be given either by multiplication by 2 on $\mathbb{R}$ or (pre)multiplication by [2] on A. In another word, there is a homotopy $h_\bullet$ between 2 and $[2]^*$ on the complex above. Then for any $f \in \bigoplus_{j=1}^{n_p} C(A^{r_{p,j}}, \mathbb{R})$ such that $df = 0$ we have $2f - [2]^* f = dh^*_{i-1}(f)$. Hence we have

$$f = \frac{1}{2}([2]^* f) + d(\frac{1}{2} h^*_{i-1}(f)).$$

Iterating this process, we have

$$f = \frac{1}{2^n}([2^n]^* f) + d(\frac{1}{2} h^*_{i-1}(f) + \frac{1}{4} h^*_{i-1}([2]^*(f)) + \cdots + \frac{1}{2^n} h^*_{i-1}([2^{n-1}]^*(f))).$$

We win by taking $n \to \infty$ since [2] is bounded by 1 and h_{i-1} is also a bounded of Banach spaces. □

Proof of Theorem 5.16. It suffices to show for any $A, B \in \mathrm{LCA}$ we have

$$R\mathrm{Hom}_{\mathrm{LCA}}(A, B) \to R\underline{\mathrm{Hom}}_{\mathrm{Cond(Ab)}}(A, B)$$

[11] In particular, this does not violate the previous warning in Remark 3.25.

is an isomorphism. By Theorem 5.9 it suffices to show the case where A is either discrete or $\mathbb{R}$ or compact. If A is discrete, then we reduce to the case $A = \mathbb{Z}$ (by classification of abelian groups), which is clear. In $A = \mathbb{R}$, then we use the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

to reduce to the case $A = \mathbb{T}$. Hence we may assume that A is compact. Now we apply $\mathbb{D}$ to the resolution

$$\bigoplus_J \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to \mathbb{D}(A) \to 0$$

and get

$$0 \to A \to \prod_I \mathbb{T} \to \prod_J \mathbb{T}$$

Hence we reduce to the case where $A = \prod_I \mathbb{T}$. On the other hand, we do the same trick for B and reduce to the case where B is either discrete or $\mathbb{R}$. Then we win by Proposition 5.17 and Remark 5.15. $\square$

5.1. **Breen-Deligne Resolution.**

The goal of this section is to prove the following theorem.

Theorem 5.19. *Let $A \in \mathrm{Ab}$. Then there is a functorial resolution of A given by*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \xrightarrow{m} \mathbb{Z}[A] \xrightarrow{\epsilon} A \to 0$$

where $m([a_1, a_2]) = [a_1 + a_2] - [a_1] - [a_2]$ and $\epsilon([a]) = a$.

As a direct consequence we have:

Corollary 5.20. *Let $A \in \mathrm{Cond}(\mathrm{Ab})$. Then there is a functorial resolution of A given by*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \xrightarrow{m} \mathbb{Z}[A] \xrightarrow{\epsilon} A \to 0.$$

Proof. Let F_n be the presheaf of abelain groups on $*_{\text{proét}}$ given by $F_n(S) = \mathbb{Z}[A^n(S)]$. Then for any $S \in *_{\text{proét}}$ we have a resolution of $A(S)$ by Theorem 5.19

$$\cdots \to F_2(S) \xrightarrow{m} F_1(S) \xrightarrow{\epsilon} A(S) \to 0.$$

By functoriality, this gives a resolution of presheaves. Since sheafification is exact, sheafification gives the desired result. $\square$

To prove the Breen-Deligne theorem, we first recall the following notion of abelian category.

Definition 5.21. Let $\mathcal{A}$ be a compactly generated abelian category which admits a set of compact projective generators $\mathcal{A}_0$. We define 0-pseudocoherent to be finitely generated. And we say $X \in \mathcal{A}$ is n-pseudocoherent with $n \geq 1$ if $R^i\mathrm{Hom}(X, -)$ commutes with filtered colimits for $0 \leq i \leq n-1$. And we say X is pseudocoherent if it is n-pseudocoherent for all n.

We first make the following observation.

Lemma 5.22. *In the setting of Definition 5.21, $X \in \mathcal{A}$ is n-pseudocoherent if and only if there exists partial resolution*

$$\bigoplus_{j=0}^{k_n} P_{n,j} \to \cdots \to \bigoplus_{j=0}^{k_0} P_{0,j} \to X \to 0$$

where $P_{i,j} \in \mathcal{A}_0$ and $k_i \geq 0$. Moreover, if X is n-pseudocoherent, then every length less than n resolution of the above form can be extended to a length n one.

Proof. The "if" part is clear. To show "only if", we use induction on n. When $n = 1$, X being 1-pseudocoherent is equivalent of being compact. Write $X \simeq \operatorname{colim} X_i$ as a filtered colimit of finitely presented objects X_i. Since X is compact, $\operatorname{Hom}(X, X) \simeq \operatorname{colim} \operatorname{Hom}(X, X_i)$ showing that X is a retract of some X_i. Hence X itself is finitely presented. Then for general n, we may assume X is n-pseudocoherent and

$$\bigoplus_{j=0}^{k_{n-1}} P_{n-1,j} \xrightarrow{f} \cdots \to \bigoplus_{j=0}^{k_0} P_{0,j} \to X \to 0$$

is constructed. It is not hard to see that $\ker(f)$ is 1-pseudocoherent. Then we win by the previous argument. The last assertion follows from the above argument and induction. ☐

Example 5.23. Let $\mathcal{A} = \operatorname{Cond}(\operatorname{Ab})$.

(1) If S is extremally disconnected, then $\mathbb{Z}[S]$ is a compact projective generator which is pseudocoherent. Moreover, for any $X \in \operatorname{CHaus}$, one can use Remark 4.5 to give a hypercover of extremally disconnected sets $S_\bullet \to X$ showing that $\mathbb{Z}[X]$ is pseudocoherent.

(2) If A is a compact abelian group, then $\mathbb{Z}[A^n]$ is pseudocoherent for each n. Hence using the Breen-Deligne resolution (Theorem 5.19) and Lemma 5.27, we know that A is pseudocoherent.

(3) Being pseudocoherent in $\mathcal{A} = \operatorname{Cond}(\operatorname{Ab})$ is equivalent of being "internal" pseudocoherent, i.e. $R^n\underline{\operatorname{Hom}}(X, -)$ commutes with filtered colimits for all n. Indeed, one can see this by noting that pseudocoherent is stable under tensor product. Then it reduces to check for $\mathbb{Z}[S]$'s where S is profinite. This is clear since $\mathbb{Z}[S] \otimes^L \mathbb{Z}[T] \simeq \mathbb{Z}[S \times T]$.

Example 5.24. The key example that will matter to the rest of this section is the following. Denote Latt as the full subcategory of lattices in Ab, i.e. finite free $\mathbb{Z}$-modules. Let $\mathcal{A} = \operatorname{Fun}(\operatorname{Latt}, \operatorname{Ab})$. Then $\mathcal{A}_0 = \{F_n \mid F_n(P) = \mathbb{Z}[P^n]\}$ is a set of compact projective generators of $\mathcal{A}$. In fact, for any $F \in \operatorname{Fun}(\operatorname{Latt}, \operatorname{Ab})$, since $P^n \simeq \operatorname{Hom}_{\operatorname{Latt}}(\mathbb{Z}^n, P)$, we have

$$\operatorname{Hom}_{\operatorname{Fun}(\operatorname{Latt},\operatorname{Ab})}(\mathbb{Z}[\operatorname{Hom}_{\operatorname{Latt}}(\mathbb{Z}^n, -)], F) \simeq \operatorname{Hom}_{\operatorname{Fun}(\operatorname{Latt},\operatorname{Set})}(\operatorname{Hom}_{\operatorname{Latt}}(\mathbb{Z}^n, -), F) \simeq F(\mathbb{Z}^n)$$

which clearly commutes with colimits and limits. Since F is determined by $F(\mathbb{Z}^n)$, $\mathcal{A}_0$ is a set of generators.

The next theorem will be the key to prove Theorem 5.19.

Theorem 5.25. *In the situation of Example 5.24, the identity functor* $\operatorname{Id}(P) = P$ *is pseudocoherent.*

Proof. We prove by induction. Clearly

$$\mathbb{Z}[P \times P] \to \mathbb{Z}[P] \to P \to 0$$

shows Id is 1-pseudocoherent. Assume it is $(n-1)$-pseudocoherent. Recall that the classifying stack BP is given by $[*/P]$, i.e.

$$\cdots P \substack{\to\\\to\\\to} P \rightrightarrows *$$

Since P is abelian, BP is again an abelain group object in Cat_∞. Repeating this process, we get B^nP. We need to compute the homology of the $\mathbb{Z}[B^nP]$ (which is the totalization of the underlying n-fold simplicial object) to proceed induction. In order to do that, we invoke the following theorem of Eilenberg-MacLane.

Theorem 5.26. *For finite free abelian group P, the homology groups*

$$H_i(\mathbb{Z}[B^nP]) = H_i(K(P,n),\mathbb{Z})$$

vanishes for $i < n$ and equals to $\pi_{i-n}(H\mathbb{Z} \otimes_{\mathbb{Z}} H\mathbb{Z}) \otimes_{\mathbb{Z}} P$ for $n \leq i < 2n$, where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum of $\mathbb{Z}$. Moreover, $\pi_{i-n}(H\mathbb{Z} \otimes_{\mathbb{Z}} H\mathbb{Z})$ is finitely generated and $\pi_0(H\mathbb{Z} \otimes_{\mathbb{Z}} H\mathbb{Z}) \simeq \mathbb{Z}$.

Let $M_j = \pi_j(H\mathbb{Z} \otimes_{\mathbb{Z}} H\mathbb{Z})$. Since $H_n(\mathbb{Z}[B^nP]) \simeq P$, we have a complex

$$\mathbb{Z}[B^nP][-n] \to P \to 0.$$

By assumption Id being $(n-1)$-pseudocoherent gives $M_j \otimes_{\mathbb{Z}} \mathrm{Id}$ being $(n-1)$-pseudocoherent. Then we win by the following lemma. □

Lemma 5.27. *Let $\mathcal{A}$, $\mathcal{A}_0$ as in Definition 5.21. Assume there is a complex*

$$C_\bullet = P_\bullet \to X \to 0$$

where P_i's are compact projective and $P_0 \to X$ is surjective. Assume further that $H_i(C_\bullet)$ is $(n-i-1)$-pseudocoherent for $0 \leq i \leq n-1$, then X is n-pseudocoherent.

Proof. Easy exercise. □

Now we prove the Breen-Deligne theorem.

Proof of Theorem 5.19. Combining Lemma 5.22 and Theorem 5.25, we have the resolution for finite free abelian groups. By Lemma 5.22, we can adjust the first three terms as in the statement. In order to extend to all abelian groups, notice that the differentials are given by universal formula, i.e. $F_n \to F_m$ is given by a finite sum of matrices acting on the base. Hence we can take filtered colimits to get the case where A is free and use simplicial resolution to get the general case.

□

6. Analytic Rings

The purpose of constructing analytic rings is to uniform different types of geometry, e.g. algebraic geometry, rigid analytic geometry, differential geometry, etc. In the framework of ringed spaces, it suffices to know the structure sheaf, or in another word, the sheaf of functions on the space. And analytic rings are such objects which characterize the "uniform" affine objects.

We first mention the following notions which will be studied in more details later.

Definition 6.1. We define the following.

(1) For a profinite set $S = \lim_i S_i$, we define
$$\mathbb{Z}[S]^{\blacksquare} := \lim_i \mathbb{Z}[S_i]$$
as a condensed abelian group. Note that there is a natural map $S = \lim_i S_i \to \lim_i \mathbb{Z}[S_i] = \mathbb{Z}[S]^{\blacksquare}$ extending to $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$.

(2) A condensed abelian group A is called solid if for any $S \in *_{\text{proét}}$ and any $f : S \to A$ as condensed sets, there is a unique lift $\mathbb{Z}[S]^{\blacksquare} \to A$. In another word,
$$\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, A) \to \mathrm{Hom}(\mathbb{Z}[S], A) = A(S)$$
is an isomorphism. We will denote the category of all solid abelian groups as Solid.

(3) A complex of condensed abelian group $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is called solid if for any $S \in *_{\text{proét}}$, we have
$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \to R\mathrm{Hom}(\mathbb{Z}[S], C) = R\Gamma(S, C)$$
is an isomorphism.

We hope to find an analytic ring $\mathbb{Z}_{\blacksquare}$ such that $\mathbb{Z}_{\blacksquare}[S] = \mathbb{Z}[S]^{\blacksquare}$ for all $S \in *_{\text{proét}}$. Here is the first attempt.

Definition 6.2. A pre-analytic ring $\mathcal{A}$ consists of the following datum:

(1) A underlying condensed ring $\underline{\mathcal{A}}$.

(2) A functor $\mathcal{A}[-]$ from the category of extremally disconnected sets to the category $\mathrm{Mod}_{\underline{\mathcal{A}}}$ of $\underline{\mathcal{A}}$-modules in Cond(Ab) which takes finite disjoint unions to finite products.

(3) A natural transformation $\alpha : \underline{\mathcal{A}}[-] \to \mathcal{A}[-]$ where $\underline{\mathcal{A}}[-] := \underline{\mathcal{A}} \otimes \mathbb{Z}[-]$.

We somtimes denote $\mathcal{A}$ as $(\underline{\mathcal{A}}, \mathcal{A}[-], \alpha)$.

Remark 6.3. Let S be a profinite set. When $\underline{\mathcal{A}}$ is discrete, $\underline{\mathcal{A}}[S]$ coincide with the sheafification of the presheaf $T \mapsto \underline{\mathcal{A}}[S(T)]$. This is because $\underline{\mathcal{A}}[-]$ is the universal functor from the category of extremally disconnected sets to $\mathrm{Mod}_{\underline{\mathcal{A}}}$ taking disjoint unions to products. In other words, both $\underline{\mathcal{A}}[-]$ and $\underline{\mathcal{A}} \otimes \mathbb{Z}[-]$ are left adjoint to the forgetful functor
$$\mathrm{Mod}_{\underline{\mathcal{A}}} \to \mathrm{Cond(Set)}.$$
Clearly, $\underline{\mathcal{A}}[-]$ is the left adjoint just as the case of $\underline{\mathcal{A}} = \mathbb{Z}$. This is verified on the presheaf level and use the fact that sheafification is left adjoint to the forgetful functor from sheaves to presheaves. To see that $\underline{\mathcal{A}} \otimes \mathbb{Z}[-]$ is the left adjoint, note that we have the following identification
$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}} \otimes \mathbb{Z}[S], M) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}, M)) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], M) \simeq M(S)$$

where $M \in \mathrm{Mod}_{\underline{\mathcal{A}}}$.

Example 6.4. We have the following examples.

(1) $\mathbb{Z}_{\blacksquare} = (\mathbb{Z}, \mathbb{Z}_{\blacksquare}[-], \alpha)$ where $\mathbb{Z}_{\blacksquare}[S] \coloneqq \mathbb{Z}[S]^{\blacksquare}$ and α is given by $\mathbb{Z}[-] \to \mathbb{Z}[-]^{\blacksquare}$.
(2) $\mathbb{Z}_{p,\blacksquare} = (\mathbb{Z}_p, \mathbb{Z}_{p,\blacksquare}[-], \alpha)$ where $\mathbb{Z}_{p,\blacksquare}[S] \coloneqq \mathbb{Z}_p[S]^{\blacksquare} = \lim_i \mathbb{Z}_p[S_i]$ and α as in (1).
(3) For a discrete ring A, we can associate it with a pre-analytic ring $(A, \mathbb{Z})_{\blacksquare} = (A, (A, \mathbb{Z})_{\blacksquare}[-], \alpha)$, where $(A, \mathbb{Z})_{\blacksquare}[S] \coloneqq \mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}} A$ and alpha is given by the one on $\mathbb{Z}_{\blacksquare}$.
(4) For a finitely generated $\mathbb{Z}$-algebra A, we can form a pre-analytic ring $A_{\blacksquare} = (A, A)_{\blacksquare} = (A, A_{\blacksquare}[-], \alpha)$, where $A_{\blacksquare}[S] = \lim_i A[S_i]$ and α similarly as above. One should note that even when A is discrete, this is different than (3).

Remark 6.5. We remark here that the underlying condensed abelian group of $\mathbb{Z}[S]^{\blacksquare}$ can be viewed more concretely. Let $C(S, \mathbb{Z})$ be the condensed abelian group $\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z})$. Since $\mathbb{Z}$ is discrete, for any $T \in *_{\text{proét}}$ we have the following

$$\begin{aligned} C(S, \mathbb{Z})(T) &= C(S \times T, \mathbb{Z})(*) \\ &\simeq \operatorname*{colim}_j \operatorname*{colim}_i C(S_i \times T_j, \mathbb{Z})(*) \\ &\simeq \operatorname*{colim}_i C(S_i \times T, \mathbb{Z})(*) \\ &\simeq \operatorname*{colim}_i C(S_i, \mathbb{Z})(T) \end{aligned}$$

showing that $C(S, \mathbb{Z}) \simeq \operatorname{colim}_i C(S_i, \mathbb{Z})$. Hence we have

$$\mathbb{Z}[S]^{\blacksquare} \simeq \lim_i \mathbb{Z}[S_i] \simeq \lim_i \underline{\mathrm{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z}) \simeq \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}).$$

Thus $\mathbb{Z}[S]^{\blacksquare}(*) \simeq \mathcal{M}(S, \mathbb{Z}) \coloneqq \mathrm{Hom}(C(S, \mathbb{Z}), \mathbb{Z})$ is the measure of S valued in $\mathbb{Z}$.

Remark 6.6. A more algebro-geometric point of view for $\mathbb{Z}[S]^{\blacksquare}$ is that it is the reflexification of $\mathbb{Z}[S]$ under the dual $\underline{\mathrm{Hom}}(-, \mathbb{Z})$.

Now we define analytic rings. The slogan for analytic rings is that their module categories are "good". We first define modules for pre-analytic rings.

Definition 6.7. Let $\mathcal{A}$ be a pre-analytic ring. Let $\mathrm{Mod}_{\mathcal{A}} \subset \mathrm{Mod}_{\underline{\mathcal{A}}}$ be the full subcategory given by M satisfying for any extremally disconnected S,

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], M) \simeq M(S)$$

is an isomorphism.

Definition 6.8. A pre-analytic ring $\mathcal{A}$ is analytic if for all bounded above complexes C of $\underline{\mathcal{A}}$-modules of the form $C_i = \bigoplus_j \mathcal{A}[T_{i,j}]$ with $T_{i,j}$ extremally disconnected, and for all extremally disconnected S,

$$R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism.

Remark 6.9. We can replace the testing spaces in Definition 6.7 and Definition 6.8 by profinite sets. However, it is usually easier to use extremally disconnected spaces since they are projective and can avoid some sheafification issues.

Example 6.10. One can easily produce a trivial analytic ring, $\mathcal{A} = (0, 0, \alpha)$.

Remark 6.11. By definition, if $\mathcal{A}$ is analytic, then $\mathcal{A}[S] \in \mathrm{Mod}_{\mathcal{A}}$ for any S extremally disconnected.

We now state the two main theorems of this section.

Theorem 6.12. *Let $\mathcal{A}$ be an analytic ring. Then we have the following for $\mathrm{Mod}_{\mathcal{A}}$.*

(1) $\mathrm{Mod}_{\mathcal{A}} \subset \mathrm{Mod}_{\underline{\mathcal{A}}}$ is stable under limits, colimits and extensions.

(2) $\{\mathcal{A}[S] \in \mathrm{Mod}_{\mathcal{A}} \mid S$ extremally disconnected$\}$ is a set of compact projective generators for $\mathrm{Mod}_{\mathcal{A}}$.

(3) The inclusion $\mathrm{Mod}_{\mathcal{A}} \subset \mathrm{Mod}_{\underline{\mathcal{A}}}$ admits a left adjoint

$$\mathrm{Mod}_{\underline{\mathcal{A}}} \to \mathrm{Mod}_{\mathcal{A}}$$

by $M \mapsto M \otimes_{\underline{\mathcal{A}}} \mathcal{A}$. Moreover, it preserves colimits and sends $\underline{\mathcal{A}}[S]$ to $\mathcal{A}[S]$.

(4) There is unique symmetric monoidal structure $- \otimes_{\mathcal{A}} -$ on $\mathrm{Mod}_{\mathcal{A}}$ such that the functor in (3) is symmetric monoidal.

Theorem 6.13. *Let $\mathcal{A}$ be an analytic ring. Let $D_{\blacksquare}(\mathrm{Mod}_{\underline{\mathcal{A}}})$ be the full subcategory of $D(\mathrm{Mod}_{\underline{\mathcal{A}}})$ given by $\{C \in D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \mid R\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \xrightarrow{\simeq} R\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)\}$. Then we have the following.*

(1) The functor

$$D(\mathrm{Mod}_{\mathcal{A}}) \to D(\mathrm{Mod}_{\underline{\mathcal{A}}})$$

is fully faithful and the essential image is $D_{\blacksquare}(\mathrm{Mod}_{\underline{\mathcal{A}}})$.

(2) We have $D_{\blacksquare}(\mathrm{Mod}_{\underline{\mathcal{A}}}) = \{C \in D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \mid H^i(C) \in \mathrm{Mod}_{\mathcal{A}}\}$.

(3) The inclusion functor by (1) admits a left adjoint

$$D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \to D(\mathrm{Mod}_{\mathcal{A}})$$

by $C \mapsto C \otimes^L_{\underline{\mathcal{A}}} \mathcal{A}$.

(4) There is unique symmetric monoidal structure $- \otimes^L_{\mathcal{A}} -$ on $D(\mathrm{Mod}_{\mathcal{A}})$ such that the functor in (3) is symmetric monoidal.[12]

Instead of proving the theorems directly, we prove the following abstract lemmas first, which will almost immediately imply the theorems.

Lemma 6.14. *Assume $\mathcal{C}$ is an abelian category with colimits and $\mathcal{C}_0$ is a full subcategory of compact projective generators of $\mathcal{C}$. Moreover, assume $F : \mathcal{C}_0 \to \mathcal{C}$ is a functor equipped with a natural transform $X \to F(X)$ satisfying the following: for any $X \in \mathcal{C}_0$ and any bounded above complex K with each K_i being a direct sum of objects in $F(\mathcal{C}_0)$,*

$$R\mathrm{Hom}(F(X), K) \xrightarrow{\simeq} R\mathrm{Hom}(X, K)$$

[12] It is not clear that $- \otimes^L_{\mathcal{A}} -$ is the left derived functor of $- \otimes_{\mathcal{A}} -$. Even though in most cases it is.

is an isomorphism. Let $\mathcal{C}_F$ *be the full subcategory with objects* $\{Y \in \mathcal{C} \mid \mathrm{Hom}(F(X), Y) \xrightarrow{\simeq} \mathrm{Hom}(X, Y)$ *for any* $X \in \mathcal{C}_0\}$*. Then we have the following*

1. $\mathcal{C}_F \subset \mathcal{C}$ *is stable under limits, colimits and extensions.*
2. $\{F(X) \in \mathcal{C} \mid X \in \mathcal{C}\}$ *is a set of compact projective generators for* $\mathcal{C}_F$*.*
3. *The inclusion* $\mathcal{C}_F \subset \mathcal{C}$ *admits a unique left adjoint*
$$L : \mathcal{C} \to \mathcal{C}_F$$
preserving colimits and extending $F : \mathcal{C}_0 \to \mathcal{C}$*.*

Lemma 6.15. *In the situation of Lemma 6.14, let* $D_F(\mathcal{C})$ *be the full subcategory of* $D(\mathcal{C})$ *given by* $\{K \in D(\mathcal{C}) \mid R\mathrm{Hom}(F(X), K) \xrightarrow{\simeq} R\mathrm{Hom}(X, K)$ *for any* $X \in \mathcal{C}_0\}$*. Then we have the following*

1. *The functor*
$$D(\mathcal{C}_F) \to D(\mathcal{C})$$
is fully faithful and the essential image is $D_F(\mathcal{C})$*.*
2. *We have* $D_F(\mathcal{C}) = \{K \in D(\mathcal{C}) \mid H^i(K) \in \mathcal{C}_F\}$*.*
3. *The inclusion functor by (1) admits a left adjoint*
$$D(\mathcal{C}) \to D_F(\mathcal{C})$$
which is the left derived functor of L *in Lemma 6.14 (3).*

We first observe that the assumption in Lemma 6.14 can be interchanged in the following sense.

Lemma 6.16. *In the situation of Lemma 6.14, for any* $X \in \mathcal{C}_0$ *and any* $K = \ker(Y \to Z)$ *where* Y, Z *are direct sums of objects in* $F(\mathcal{C}_0)$*, we have*
$$R\mathrm{Hom}(F(X), K) \xrightarrow{\simeq} R\mathrm{Hom}(X, K)$$
is an isomorphism.

As a direct consequence, by taking truncations, we get the following corollary.

Corollary 6.17. *In the situation of Lemma 6.14, for any* $X \in \mathcal{C}_0$ *and any* K *where each* K_i *is a direct sum of objects in* $F(\mathcal{C}_0)$*, we have*
$$R\mathrm{Hom}(F(X), K) \xrightarrow{\simeq} R\mathrm{Hom}(X, K)$$
is an isomorphism. In particular, the conditions in Lemma 6.14 and Lemma 6.16 are equivalent.

Remark 6.18. By Lemma 6.16, for the definition of analytic rings, we can consider $K = \ker(Y \to Z)$, where Y, Z are direct sums of objects in $F(\mathcal{C}_0)$, as test objects instead of the bounded complexes.

Proof of Lemma 6.16. Choose $B_\bullet$ be a resolution of K where $B_i = \bigoplus_j X_{i,j}$ with $X_{i,j} \in \mathcal{C}_0$. Let $C_i = \bigoplus_j F(X_{i,j})$. Then $C_\bullet$ is a complex with a natural map $B \to C$. Note that the assumption of Lemma 6.14 implies that $R\mathrm{Hom}(F(X_{i,j}), Y) \xrightarrow{\simeq} R\mathrm{Hom}(X_{i,j}, Y)$ is an isomorphism. Hence
$$R\mathrm{Hom}(C, Y) \xrightarrow{\simeq} R\mathrm{Hom}(B, Y)$$

is an isomorphism. Similarly, we have $R\mathrm{Hom}(C,Z) \xrightarrow{\simeq} R\mathrm{Hom}(B,Z)$. Since $K = \ker(Y \to Z)$, the above discussion implies that there is a unique map $C_\bullet \to K$ extends the following diagram.

$$\begin{array}{ccccccc} B_\bullet & \longrightarrow & K & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\ C_\bullet & \dashrightarrow & K & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

In particular, since $B_\bullet \to K$ is a quasi-isomorphism in $D(\mathcal{C})$, $B_\bullet$ is a retract of $C_\bullet$. Thus we get

$$R\mathrm{Hom}(F(X), B_\bullet) \xrightarrow{\simeq} R\mathrm{Hom}(X, B_\bullet)$$

by the assumption of 6.14. □

We will prove Lemma 6.14 and Lemma 6.15 together.

Proof of Lemma 6.14 and Lemma 6.15. Clearly $\mathcal{C}_F \subset \mathcal{C}$ is stable under limits. We first show that it is also stable under cokernels. Assume $Y, Z \subset \mathcal{C}_F$ and $Q = \mathrm{coker}(Y \to Z)$. Resolve Z by $\bigoplus_i P_i$ with $P_i \in \mathcal{C}_0$. Since $Z \in \mathcal{C}_F$, this resolution uniquely extends to a resolution $\bigoplus_i F(P_i) \to Z$. Do the same trick for the fibre product $W = Y \times_Z (\bigoplus_i F(P_i))$ and get a resolution $\bigoplus_j F(Q_j)$. In conclusion, we get the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_3 & \longrightarrow & \bigoplus_j F(Q_j) & \longrightarrow & \bigoplus_i F(P_i) & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \downarrow{=} & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & W & \longrightarrow & \bigoplus_i F(P_i) & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \ulcorner & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & K_1 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

As a consequence we may assume Y, Z are direct sums of objects in $\mathcal{C}_0$. Hence for both Y, Z, K_3 the map $R\mathrm{Hom}(F(X), -) \to R\mathrm{Hom}(X, -)$ are isomorphism for all $X \in \mathcal{C}_0$ giving the desired isomorphism

$$R\mathrm{Hom}(F(X), Q) \to R\mathrm{Hom}(X, Q)$$

for Q. The above argument actually shows the following two consequences:

(a) $\mathcal{C}_F$ are exactly the objects of the form $\mathrm{coker}(\bigoplus_j F(Q_j) \to \bigoplus_i F(P_i))$ where $Q_j, P_i \in \mathcal{C}_0$.

(b) $Q[0] \in D_F(\mathcal{C})$ for all $Q \in \mathcal{C}_F$.

(a) implies that $\mathcal{C}_F$ is stable under direct sums and extensions. Hence stable under colimits (using colimits are cokernels of direct sums) and extensions proving Lemma 6.14 (1). This also implies Lemma 6.14 (2) and (3) immediately.

We now prove Lemma 6.15. To see fully faithfulness of $D(\mathcal{C}_F) \to D(\mathcal{C})$, observe the following

$$R\mathrm{Hom}_{\mathcal{C}_F}(F(X), Q) \xrightarrow{\simeq} R\mathrm{Hom}_{\mathcal{C}}(X, Q)$$

for any $Q \in \mathcal{C}_F$ and $X \in \mathcal{C}_0$. In fact, the isomorphism of $R^0\mathrm{Hom}$ is by the definition of $\mathcal{C}_F$. As for $i > 0$, note that X is projective in $\mathcal{C}$ and $F(X)$ is projective in $\mathcal{C}_F$ (by Lemma 6.14). Hence both sides vanishes. Then by passing to Postnikov limit and truncations $\tau^{\leq i}$ we see that

$$R\mathrm{Hom}_{\mathcal{C}_F}(F(X), K) \xrightarrow{\simeq} R\mathrm{Hom}_{\mathcal{C}}(X, K)$$

holds for any $K \in D(\mathcal{C}_F)$. Therefore, we have a fully faithful functor

$$D(\mathcal{C}_F) \to D(\mathcal{C})$$

as desired. To describe the essential image, let $D'_F(\mathcal{C})$ denote the full subcategory $\{K \in D(\mathcal{C}) \mid H^i(K) \in \mathcal{C}_F\}$. We first see that $D'_F(\mathcal{C}) \subset D_F(\mathcal{C})$. In fact, since $\mathcal{C}_F$ is stable under limits and colimits, $D'_F(\mathcal{C})$ is stable under direct sums and products. Hence by passing to Postnikov limit and truncations $\tau^{\leq i}$ again, we may assume that $K = X[0]$ with $X \in \mathcal{C}_F$. However, this is already established by (b) and the isomorphism above. Since $D_F(\mathcal{C})$ is generated by $F(X)[0]$ with $X \in \mathcal{C}$ under shifts and colimits, we see that $D'_F(\mathcal{C}) = D_F(\mathcal{C})$. To show that $D(\mathcal{C}_F) \to D'_F(\mathcal{C}) = D_F(\mathcal{C})$ is essential surjective, note that it is suffices to show it on hearts (as we already established fully faithful and this functor commutes with direct sums and products). Now Lemma 6.15 (1) and (2) are done. The existence the functor in (3) is formal. To check it is the left derived of $L : \mathcal{C} \to \mathcal{C}_F$, it suffices to check on $\mathcal{C}_0$ which is clear. □

We now prove the two main theorems.

Proof of Theorem 6.12 and Theorem 6.13. Let $\mathcal{C} = \mathrm{Mod}_{\underline{\mathcal{A}}}$ and $\mathcal{C}_0$ be the full subcategory of $\underline{\mathcal{A}}[S]$ for S extremally disconnected. Also, set $F(\underline{\mathcal{A}}[S]) = \mathcal{A}[S]$. Then by Lemma 6.14 and Lemma 6.15, it remains to prove that both $\mathrm{Mod}_{\mathcal{A}}$ and $D(\mathrm{Mod}_{\mathcal{A}})$ admits a unique symmetric monoidal structure that satisfies the required condition. The strategy is similar, we just show for $D(\mathrm{Mod}_{\mathcal{A}})$. We claim that the kernel of $D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \to D(\mathrm{Mod}_{\mathcal{A}})$ is stable under $- \otimes^L_{\underline{\mathcal{A}}} -$. Then we can endow $D(\mathrm{Mod}_{\mathcal{A}})$ with the symmetric monoidal structure given by the Bousfield localization. To prove the claim, notice that

$$R\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S] \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{A}}[T], K) \xrightarrow{\simeq} R\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S \times T], K)$$

for any extremally disconnected sets S, T and $K \in D(\mathrm{Mod}_{\mathcal{A}})$. □

Remark 6.19. From now on, when there is no confusion, we will just denote both $D(\mathrm{Mod}_{\mathcal{A}})$ and $D_{\blacksquare}(\mathrm{Mod}_{\underline{\mathcal{A}}})$ as $D(\mathcal{A})$ for short.

We now define morphisms between analytic rings and how to base change.

Definition 6.20. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-analytic rings. Then a morphism $f : \mathcal{A} \to \mathcal{B}$ consists of the following

(1) A morphism $f : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ of condensed rings.
(2) A natural transformation $\mathcal{A}[-] \to \mathcal{B}[-]$ which is compatible with $\underline{\mathcal{A}}[-] \to \mathcal{A}[-]$ and $\underline{\mathcal{B}}[-] \to \mathcal{B}[-]$.

Definition 6.21. Let $\mathcal{A}$ and $\mathcal{B}$ be analytic rings. Then a morphism $f : \mathcal{A} \to \mathcal{B}$ consists of the following datum:

(1) A morphism $f : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ of condensed rings.

(2) For any S extremally disconnected, $\mathcal{B}[S]$ as an $\underline{\mathcal{A}}$-module lies in the subcategory $\mathrm{Mod}_{\mathcal{A}}$. Consequently, the map $S \to \mathcal{B}[S]$ extends uniquely to $\mathcal{A}[S] \to \mathcal{B}[S]$.

Remark 6.22. It is not clear that a map of pre-analytic rings between analytic rings is analytic. But this is true in most of the practical cases, c.f. [SCa, Proposition 7.14, Remark 7.15].

We now define the base change functor between analytic rings.

Proposition 6.23. *Let $\mathcal{A} \to \mathcal{B}$ be a map of analytic rings. Then we have the following.*

(1) There is a unique functor from $\mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\mathcal{B}}$ such that the following diagram commutes.

$$\begin{CD} \mathrm{Mod}_{\mathcal{A}} @>{-\otimes_{\mathcal{A}}\mathcal{B}}>> \mathrm{Mod}_{\mathcal{B}} \\ @A{-\otimes_{\underline{\mathcal{A}}}\mathcal{A}}AA @A{-\otimes_{\underline{\mathcal{B}}}\mathcal{B}}AA \\ \mathrm{Mod}_{\underline{\mathcal{A}}} @>{-\otimes_{\underline{\mathcal{A}}}\underline{\mathcal{B}}}>> \mathrm{Mod}_{\underline{\mathcal{B}}} \end{CD}$$

We denote this functor as $-\otimes_{\mathcal{A}} \mathcal{B}$.

(2) There is a unique functor from $D(\mathrm{Mod}_{\mathcal{A}}) \to D(\mathrm{Mod}_{\mathcal{B}})$ such that the following diagram commutes.

$$\begin{CD} D(\mathrm{Mod}_{\mathcal{A}}) @>{-\otimes^L_{\mathcal{A}}\mathcal{B}}>> D(\mathrm{Mod}_{\mathcal{B}}) \\ @A{-\otimes^L_{\underline{\mathcal{A}}}\mathcal{A}}AA @A{-\otimes^L_{\underline{\mathcal{B}}}\mathcal{B}}AA \\ D(\mathrm{Mod}_{\underline{\mathcal{A}}}) @>{-\otimes^L_{\underline{\mathcal{A}}}\underline{\mathcal{B}}}>> D(\mathrm{Mod}_{\underline{\mathcal{B}}}) \end{CD}$$

We denote this functor as $-\otimes^L_{\mathcal{A}} \mathcal{B}$.

Proof. For (1), it suffices to show on the generators. However, this is obvious that $\mathcal{B}[S] \simeq \mathcal{A}[S] \otimes \mathcal{B}$ since $\underline{\mathcal{B}}[S] \simeq \underline{\mathcal{A}}[S] \otimes \underline{\mathcal{B}}$ and the symmetric monoidal structure on $\mathrm{Mod}_{\mathcal{A}}$ (resp. $\mathrm{Mod}_{\mathcal{B}}$) comes from $\mathrm{Mod}_{\underline{\mathcal{A}}}$ (resp. $\mathrm{Mod}_{\underline{\mathcal{B}}}$). As for (2), one notice that, by reducing to cohomology, if $C \in D(\mathrm{Mod}_{\mathcal{B}})$ then as a object in $D(\mathrm{Mod}_{\underline{\mathcal{A}}})$ it lies in $D(\mathrm{Mod}_{\mathcal{A}})$. Then we win by the similar reason as (1). In this case, $-\otimes^L_{\mathcal{A}} \mathcal{B}$ must be a left derived functor as it commutes with colimits and preserves compact projective generators $\mathcal{A}[S] \mapsto \mathcal{B}[S]$. ☐

6.1. **Animation.** We follow the path of [SCc, 11.1] and [Lur06, 5.5.8] to recall the notion of animation. Then we can generalize all the above discussion to the animated setting. One of the advantages of this generalization is to get a converse of Theorem 6.12 and Theorem 6.13.

We start by recalling some general notions from ∞-category theory. We start with the notion of cofinality from [Lur06, Definition 4.1.1.1].

Definition 6.24. A map $f : K \to L$ of simplicial sets is called cofinal if for any right fibration (e.g. a Kan fibration) $M \to L$, the induced map of simplicial sets

$$\mathrm{Hom}_L(L, M) \to \mathrm{Hom}_L(K, M)$$

is a homotopy equivalence.

Cofinal maps describe exactly when colimits coincide by [Lur06, Proposition 4.1.1.8]:

Proposition 6.25. *Let $v : K' \to K$ be a map of simplicial sets. Then v is cofinal if and only if the following equivalent condition holds.*

(1) For any ∞-category $\mathcal{C}$ and any diagram $p : K \to \mathcal{C}$, the induced map $\mathcal{C}_{p/} \to \mathcal{C}_{p\circ v/}$ is an equivalence.

(2) For any ∞-category $\mathcal{C}$ and any diagram $\tilde{p} : K^{\triangleright} \to \mathcal{C}$ which is a colimit of $\tilde{p}|_K$, the induced diagram $\tilde{p}' : K'^{\triangleright} \to \mathcal{C}$ is a colimit of $\tilde{p}'|_{K'}$.

We then recall the notion of filtered ∞-categories and colimits from [Lur06, 5.3.1].

Definition 6.26. Let $\mathcal{C}$ be an ∞-category and κ is a regular cardinal. Then $\mathcal{C}$ is κ-filtered if for every κ-small simplicial set K and every map $f : K \to \mathcal{C}$, there is a lift to $\tilde{f} : K^{\triangleright} \to \mathcal{C}$. As before, we drop the prescript κ if $\kappa = \aleph_0$. A κ-filtered colimit is the one indexed by a κ-filtered ∞-category.

Hence combining Proposition 6.25 and the following proposition from [Lur06, Proposition 5.3.1.18], in terms of filtered colimits it is equivalent of the classical sense:

Proposition 6.27. *Let $\mathcal{C}$ be a κ-filtered ∞-category. Then there is a κ-filtered partially ordered set A and a cofinal map $N(A) \to \mathcal{C}$.*

Note that filtered colimits are exactly those colimits which commute with finite limit (c.f. [Lur06, 5.3.2]). One can generalize filtered colimits with the notion of sifted colimits which commute with finite products. We first give an initial definition in the sense of [Lur06, Definition 5.5.8.1].

Definition 6.28. A nonempty simplicial set K is sifted if the diagonal map $K \to J \times K$ is cofinal. In particular, a colimit is sifted if it is indexed by a sifted simplicial set.

Example 6.29. We give two essential classes of sifted simplicial sets.

(1) Filtered ∞-categories are sifted by [Lur06, Proposition 5.3.1.22]. In fact, an ∞-category $\mathcal{C}$ is κ-filtered if and only if, for each κ-small simplicial set K, the diagonal map $\delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is cofinal.

(2) Δ^{op} is sifted by [Lur06, Lemma 5.5.8.4]. Indeed, by Joyal's generalization of Quillen's Theorem A ([Lur06, Theorem 4.1.3.1]), it suffices to show that for every object $([m], [n]) \in \Delta \times \Delta$ the category $\Delta_{/[m]} \times_{\Delta} \Delta_{/[n]}$ has weakly contractible nerve. This reduces to show that the barycentric subdivision of $\Delta^m \times \Delta^n$ is weakly homotopy equivalent to $\Delta^m \times \Delta^n$ which is true.

[Lur06, Proposition 5.5.8.6] shows that sifted colimits commute with finite products.

Proposition 6.30. *Let K be a sifted simplicial set and $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be ∞-categories admitting K-indexed colimits. Suppose $f : \mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a map which preserved K-indexed colimits separately in each variable. Then f preserves K-indexed colimits.*

Our goal is to show that under mild conditions, freely adjoining sifted colimits are equivalent to freely adjoining filtered colimits and geometric realizations. Recall the notion of geometric realizations from [Lur06, Notation 6.1.2.12].

Definition 6.31. Let $\mathcal{C}$ be an ∞-category. Then the geometric realization of a simplicial diagram $U_\bullet : N(\Delta)^{\mathrm{op}} \to \mathcal{C}$ is its colimit (if sucha a colimit exists) and we denote it as $|U_\bullet|$.

Remark 6.32. If $\mathcal{C}$ is a n-category, then geometric realizations are equivalent to the colimits of their 1-truncations $N(\Delta_{\leq n})^{\mathrm{op}} \to \mathcal{C}$. When $n = 1$, it is equivalent the reflexive coequalizer, i.e. coequalizer of the parallel arrows $Y \rightrightarrows Z$ with a common section $Z \to Y$, c.f. [Lur17, Lemma 1.3.3.10]. Indeed, let $V_\bullet$ be the left Kan extension of the restriction $U_{\leq n} : N(\Delta_{\leq n})^{\mathrm{op}} \to \mathcal{C}$. We wish to prove that $\alpha_\bullet : V_\bullet \to U_\bullet$ have the same essential image in $\mathrm{PSh}(\mathcal{C})$. By [Lur06, Remark 5.3.5.6], the Yoneda embedding $\mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ factors through $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \tau_{\leq n-1}\mathcal{S})$, hence it suffices to show that $\tau_{\leq n-1}|\alpha_\bullet|$ induces an equivalence in $\mathrm{PSh}(\mathcal{C})$. Then it suffices to show that $|\alpha_\bullet|$ is n-connective. Then we win by [Lur06, Lemma 6.5.3.10].

Definition 6.33. Let $\mathcal{C}$ be an ∞-category and $C \in \mathcal{C}$.

(1) We say C is κ-compact if the functor $j_C : \mathcal{C} \to \mathcal{S}$ corepresented by C preserves κ-filtered colimits.

(2) We say C is projective if the functor $j_C : \mathcal{C} \to \mathcal{S}$ corepresented by C preserves geometric realizations.

We denote $\mathcal{C}^{\mathrm{cp}}$ as the full subcategory of $\mathcal{C}$ spanned by compact projective objects.

Remark 6.34. Granting Proposition 6.40, it is not hard to see that compact projective objects are exactly those preserving sifted colimits, c.f. [Lur06, Proposition 5.5.8.25].

Remark 6.35. Let $\mathcal{A}$ be an abelian category. Then $P \in \mathcal{A}$ is compact if and only if it is compact regarding as an object of $N(\mathcal{A})$ by Proposition 6.27. Similarly, P is projective if and only if it is projective regarding as an object of $N(\mathcal{A})$. In fact, by the previous remark, $\mathrm{Hom}(P, -)$ preserves reflexive coequalizers. In particular, it preserves effective epimorphisms. Hence, this is equivalent of preserving quotient.

Example 6.36. We have the following basic examples.

(1) If $\mathcal{C} = \mathrm{Set}$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of finite sets, which generates $\mathcal{C}$ under small colimits.

(2) If $\mathcal{C} = \mathrm{Ab}$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of finite free abelian groups, which generates $\mathcal{C}$ under small colimits.

(3) If $\mathcal{C} = \mathrm{Ring}$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of retracts of polynomial rings $\mathbb{Z}[X_1, \ldots, X_n]$, which generates $\mathcal{C}$ under small colimits.

The following examples are the condensed enhancement of the above ones.

Example 6.37. We have the following examples of condensed objects.

(1) If $\mathcal{C} = \mathrm{Cond}(\mathrm{Set})$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of extremally disconnected sets, which generates $\mathcal{C}$ under small colimits.

(2) If $\mathcal{C} = \mathrm{Cond}(\mathrm{Ab})$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of direct summands of $\mathbb{Z}[S]$ for extremally disconnected S, which generates $\mathcal{C}$ under small colimits.

(3) If $\mathcal{C} = \mathrm{Cond}(\mathrm{Ring})$, then $\mathcal{C}^{\mathrm{cp}}$ is the category of retracts of $\mathbb{Z}[\mathbb{N}[S]]$ for extremally disconnected S, where $\mathbb{N}[S]$ is the free condensed abelian monoid on S. Again we have $\mathcal{C}^{\mathrm{cp}}$ generates $\mathcal{C}$ under small colimits.

We have already seen (1) in Remark 2.16. For (2) and (3), just observe that the forgetful functor $\mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Set})$ (resp. $\mathrm{Cond}(\mathrm{Ring}) \to \mathrm{Cond}(\mathrm{Set})$) admits left adjoint $S \to \mathbb{Z}[S]$ (resp. $S \to \mathbb{Z}[\mathbb{N}[S]]$). Hence by adjointness one easily finds compact project objects in Cond(Ab) and Cond(Ring) as claimed.

Example 6.38. Example 6.37 has a more general phenomenon. Namely, if a 1-category $\mathcal{C}$ is generated by $\mathcal{C}^{\mathrm{cp}}$ under small colimits then $\mathrm{Cond}(\mathcal{C})$ is also generated under small colimits by its compact projective objects. Fix a strong limit cardinal κ. Observe that for any extremally disconnected set S, the functor

$$\mathrm{Cond}_\kappa(\mathcal{C}) \to \mathcal{C}$$

via $M \mapsto M(S)$ admits a left adjoint $X \mapsto X[S]$. This is the sheafification of the presheaf $T \mapsto \bigsqcup_{C(T,S)} X$. It is formal to check that this is the left adjoint. Then it is easy to see that $X[S]$ is the compact projective generators for $\mathrm{Cond}(\mathcal{C})$ for S extremally disconnected and $X \in \mathcal{C}^{\mathrm{cp}}$.

The following is [Lur06, Definition 5.5.8.8].

Notation 6.39. Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. We denote $\mathrm{PSh}_\Sigma(\mathcal{C})$ be the full subcategory of $\mathrm{PSh}(\mathcal{C})$ spanned by those functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ preserving finite products.

We summarize the following features of $\mathrm{PSh}_\Sigma(\mathcal{C})$ from [Lur06, Lemma 5.5.8.14, Proposition 5.5.8.15] which says that $\mathrm{PSh}_\Sigma(\mathcal{C})$ is freely generated by sifted colimits. Moreover, it is freely generated by filtered colimits and geometric realizations.

Proposition 6.40. *Let $\mathcal{C}$ be small ∞-category which admits finite coproducts. Then $\mathrm{PSh}_\Sigma(\mathcal{C})$ is stable under sifted colimits and for any $X \in \mathrm{PSh}(\mathcal{C})$, X belongs to $\mathrm{PSh}_\Sigma(\mathcal{C})$ if and only if there is a simplicial diagram $U_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{C})$ whose colimit in $\mathrm{PSh}(\mathcal{C})$ is X.*

Moreover, $\mathrm{PSh}_\Sigma(\mathcal{C})$ satisfies the following universal property. Let $\mathcal{D}$ be an ∞-category which admits filtered colimits and geometric realizations. Let $\mathrm{Fun}_\Sigma(\mathrm{PSh}_\Sigma(\mathcal{C}), \mathcal{D})$ be the full subcategory of functors preserving filtered colimits and geometric realizations. Then we have

(1) The Yoneda embedding $j : \mathcal{C} \to \mathrm{PSh}_\Sigma(\mathcal{C})$ induces an equivalence

$$\theta : \mathrm{Fun}_\Sigma(\mathrm{PSh}_\Sigma(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

(2) Any functor $F \in \mathrm{Fun}_\Sigma(\mathrm{PSh}_\Sigma(\mathcal{C}), \mathcal{D})$ preserves sifted colimits.

(3) Assume that $\mathcal{D}$ admits finite coproducts. A functor $F \in \mathrm{Fun}_\Sigma(\mathrm{PSh}_\Sigma(\mathcal{C}), \mathcal{D})$ preserves small colimits if and only if $F \circ j$ preserves finite coproducts.

Remark 6.41. Due to the above Proposition, in [SCc], $\mathrm{PSh}_\Sigma(-)$ is denoted as $\mathrm{sInd}(-)$ to capture the feature that $\mathrm{PSh}_\Sigma(-)$ is freely generated by sifted colimits. Hence if $\mathcal{C}$ is generated by compact projective objects under small colimits, then the left derived functor of the inclusion $\mathrm{PSh}_\Sigma(\mathcal{C}^{\mathrm{cp}}) \to \mathcal{C}$ is an equivalence of ∞-categories, c.f. [Lur06, Proposition 5.5.8.25].

We now come to the definition of animation.

Definition 6.42. Let $\mathcal{C}$ be a 1-category that admits small colimits and is generated by $\mathcal{C}^{\mathrm{cp}}$ under small colimits. Then we define the animation $\mathrm{Ani}(\mathcal{C})$ of $\mathcal{C}$ to be the ∞-category freely generated sifted colimits, i.e.

$$\mathrm{Ani}(\mathcal{C}) := \mathrm{PSh}_\Sigma(N(\mathcal{C}^{\mathrm{cp}})) = \mathrm{Fun}_\Sigma(N(\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathcal{S}).$$

Example 6.43. The most essential example is the ∞-category of spaces. $\mathcal{S}^{\mathrm{cp}}$ are exactly those homotopy equivalent to finite sets. Hence we have $\mathcal{S} = \mathrm{Ani}(\mathrm{Set})$. We will denote it as Ani, a.k.a the ∞-category of anima. For general properties of Kan complexes, we refer to [Ker, Chapter 3].

We now come back to the previous examples: Example 6.36 and Example 6.37.

Example 6.44. The following animations correspond to Example 6.36.

(1) If $\mathcal{C} = \mathrm{Set}$, then $\mathrm{Ani}(\mathcal{C}) = \mathrm{Ani} = \mathcal{S}$ as remarked above.

(2) If $\mathcal{C} = \mathrm{Ab}$, then $\mathrm{Ani}(\mathrm{Ab})$ is equivalent to the ∞-derived category of abelian groups $\mathcal{D}_{\geq 0}(\mathrm{Ab})$ in nonnegative homological degrees.

(3) If $\mathcal{C} = \mathrm{Ring}$, then one gets the ∞-category of animated rings. It is the homotopy coherent nerve of the fibrant-cofibrant objects of simplicial (commutative) rings. Equivalently, one can also take ∞-category of the simplicial model category of simplicial rings. This is by [Lur06, Corollary 5.5.9.3].

We want to describe the animation of condensed objects. We start with defining condensed ∞-categories.

Notation 6.45. Let $\mathcal{C}$ be an ∞-category that admits all small colimits. For any uncountable strong limit cardinal κ, we define the ∞-category $\mathrm{Cond}_\kappa(\mathcal{C})$ of κ-condensed objects of $\mathcal{C}$ be the category of contravariant functors from κ-small extremally disconnected sets to $\mathcal{C}$ taking finite coproducts to products. As before, using the fully faithful left adjoints to the forgetful functors, we define

$$\mathrm{Cond}(\mathcal{C}) \coloneqq \operatorname*{colim}_{\kappa} \mathrm{Cond}_\kappa(\mathcal{C}).$$

The following lemma describes the nature of animated condensed objects.

Lemma 6.46. *If $\mathcal{C}$ is a 1-category generated by compact projective objects under small colimits, then there is a natural equivalence of ∞-categories*

$$\mathrm{Cond}(\mathrm{Ani}(\mathcal{C})) \simeq \mathrm{Ani}(\mathrm{Cond}(\mathcal{C})).$$

Proof. Similar as in Example 6.38, both sides are generated by $X[S]$ for $X \in \mathcal{C}^{\mathrm{cp}}$ (viewing as an object of $\mathrm{PSh}(N(\mathcal{C}^{\mathrm{cp}}))$) and S extremally disconnected under small colimits. □

Now we give the definition of analytic animated associated ring.

Definition 6.47. A pre-analytic animated associative ring $\mathcal{A}$ consists of the following datum:

(1) A underlying condensed animated associative ring $\underline{\mathcal{A}}$.

(2) A functor $\mathcal{A}[-]$ from the category of extremally disconnected sets to the category $\mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ of condensed animated $\underline{\mathcal{A}}$-modules which takes finite disjoint unions to finite products.

(3) A natural transformation $\alpha : [-] \to \mathcal{A}[-]$ of condensed anima.

We somtimes denote $\mathcal{A}$ as $(\underline{\mathcal{A}}, \mathcal{A}[-], \alpha)$.

Definition 6.48. An analytic animated associative ring $\mathcal{A}$ is a pre-analytic animated associative ring such that for any $C \in \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ which is a sifted colimit of objects of the form $\mathcal{A}[S]$ for S extremally disconnected, the natural morphism

$$\underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathcal{A}[T], C) \to \underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[T], C)$$

is an equivalence for all extremally disconnected set T. We denote $\mathcal{D}_{\geq 0}(\mathcal{A})$ the full subcategory of $\mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ spanned by the objects C such that the natural morphism

$$\underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathcal{A}[T], C) \to \underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[T], C)$$

is an equivalence for all extremally disconnected set T.

We say $\mathcal{A}$ is complete (or normalized) if the natural map $\underline{\mathcal{A}} \to \mathcal{A}[*]$ is an equivalence.

We now give the same result as the "non-animated" analytic ring.

Proposition 6.49. *Let $\mathcal{A}$ be an analytic animated associated ring. Then we have the following for $\mathcal{D}_{\geq 0}(\mathcal{A})$.*

(1) $\mathcal{D}_{\geq 0}(\mathcal{A}) \subset \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ is stable under small limits and colimits.

(2) The ∞-category $\mathcal{D}_{\geq 0}(\mathcal{A})$ is generated by $\mathcal{A}[S]$ for S extremally disconnected under sifted colimits.

(3) The inclusion $\mathcal{D}_{\geq 0}(\mathcal{A}) \subset \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ admits a left adjoint

$$\mathcal{D}_{\geq 0}(\underline{\mathcal{A}}) \to \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$$

sending $\underline{\mathcal{A}}[S]$ to $\mathcal{A}[S]$ which denoted as $- \otimes^{L}_{\underline{\mathcal{A}}} \mathcal{A}$.

Remark 6.50. From Proposition 6.49, giving an animated analytic ring is equivalent of giving an condensed animated associated ring $\underline{\mathcal{A}}$ and a full subcategory $\mathcal{D}_{\geq 0}(\mathcal{A}) \subset \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ satisfying the following properties:

(1) $\mathcal{D}_{\geq 0}(\mathcal{A})$ is stable under limits and colimits.

(2) $\mathcal{D}_{\geq 0}(\mathcal{A})$ is stable under $\underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[S], -)$ for all extremally disconnected set S.

(3) The inclusion $\mathcal{D}_{\geq 0}(\mathcal{A}) \subset \mathcal{D}_{\geq 0}(\underline{\mathcal{A}})$ has a left adjoint which will be denoted as $- \otimes^{L}_{\underline{\mathcal{A}}} \mathcal{A}$.

The forward direction is garanteed by Proposition 6.49. Conversely, one just need to define $\mathcal{A}[S] \coloneqq \underline{\mathcal{A}}[S] \otimes^{L}_{\underline{\mathcal{A}}} \mathcal{A}$ for all extremally disconnected S, i.e. $\mathcal{A}[S]$ is the image of $\underline{\mathcal{A}}[S]$ under the left adjoint of the inclusion.

Remark 6.51. Passing to spectrum objects, all the above holds for $\mathcal{D}$ instead of just $\mathcal{D}_{\geq 0}$.

7. Solid Abelian Groups

The goal is to prove that $\mathbb{Z}_\blacksquare$ is an analytic ring. Recall from Example 6.4 that $\mathbb{Z}_\blacksquare$ represents the pre-analytic ring $(\mathbb{Z}, \mathbb{Z}_\blacksquare[-], \alpha)$ where $\mathbb{Z}_\blacksquare[S] := \mathbb{Z}[S]^\blacksquare$ and α is given by $\mathbb{Z}[-] \to \mathbb{Z}[-]^\blacksquare$. And as we observed in Remark 6.5, $\mathbb{Z}_\blacksquare[-] = \mathcal{M}(-, \mathbb{Z})$ is the "$\mathbb{Z}$-valued Haar measure". Hence $\mathbb{Z}_\blacksquare$ being an analytic has the following consequence: for any profinite set S, $f : S \to A$ with $A \in \mathrm{Mod}_{\mathbb{Z}_\blacksquare}$ and $\mu \in \mathcal{M}(S, \mathbb{Z})$, we can "integrate" f with respect to μ:

$$\int f d\mu \in A.$$

by $\mathrm{Hom}(\mathbb{Z}[S]^\blacksquare, A) \xrightarrow{\simeq} \mathrm{Hom}(\mathbb{Z}[S], A) = A(S)$. Now we state the main theorem.

Theorem 7.1. *$\mathbb{Z}_\blacksquare$ is an analytic ring. More specifically, for all bounded above complexes C whose terms are of the form $\bigoplus_i \mathbb{Z}_\blacksquare[T_i]$ with T_i extremally disconnected,*

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S], C) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

is an isomorphism.

Remark 7.2. One important example is when $S = \mathbb{N} \cup \{\infty\}$. This is given by $[n+1] \to [n]$ sending the $i \mapsto i$ for $i \le n$ and $i = n+1 \mapsto n$. We identify $(x_k) \in S$ with $\{n\}$ if $x_k = n$ for k sufficiently large and $(\dots, 2, 1, 0)$ as $\{\infty\}$. We define $\mathrm{NS}(\mathbb{Z}) := \mathbb{Z}[S]/(\infty)$ to be the space of nullsequence. And let $\mathrm{NS}_\blacksquare(\mathbb{Z}) := \mathbb{Z}_\blacksquare[S]/(\infty) \simeq \mathbb{Z}[[T]]$ be the space of formal sums. In fact,

$$\mathbb{Z}_\blacksquare[S]/(\infty) = \lim \mathbb{Z}[\{0, \dots, n-1, \infty\}]/(\infty) \simeq \lim \mathbb{Z}[T]/T^n \simeq \mathbb{Z}[[T]].$$

However, this is slightly abusing notations since the convergent sequence can be anything, not necessarily $T, T^2, \dots$. Another interpretation is to think $\mathrm{NS}_\blacksquare(\mathbb{Z})$ as the monoid on characters which sends ∞ to 0. Then for any nullsequence $(x_0, x_1, \dots) : \mathrm{NS}(\mathbb{Z}) \to C$ and any sequence of integers $(a_0, a_1, \dots) \in \mathrm{NS}_\blacksquare(\mathbb{Z})$, the infinite sum $\sum_n a_n x_n \in C$ converges.

Let us state some consequences of Theorem 7.1 due to Theorem 6.12 and Theorem 6.13.

Proposition 7.3. *We have the following for* Solid.

(1) Solid $\subset$ Cond(Ab) *is stable under limits, colimits and extensions.*

(2) $\{\mathbb{Z}_\blacksquare[S] \mid S$ *extremely disconnect.*$\}$ *are the compact projective generators.*

(3) The inclusion Solid $\subset$ Cond(Ab) *admits a left adjoint*

$$\mathrm{Cond(Ab)} \to \mathrm{Solid}$$

by $M \mapsto M^\blacksquare := M \otimes_\mathbb{Z} \mathbb{Z}_\blacksquare$. It preserves colimits and sends $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^\blacksquare$.

(4) There is a unique symmetric monoidal structure $- \otimes^\blacksquare - := - \otimes_{\mathbb{Z}_\blacksquare} -$ on Solid *such that the functor in (3) is symmetric monoidal.*

Moreover, we have the following derived enhancement.

(1') $D(\mathrm{Solid})$ is equivalent to the full subcategory of solid complexes in $D(\mathrm{Cond(Ab)})$ in the sense of Definition 6.1 (3).

(2') A complex is solid if and only every cohomology is solid.

(3') The inclusion $D(\mathrm{Solid}) \subset D(\mathrm{Cond}(\mathrm{Ab}))$ *has a left adjoint*

$$D(\mathrm{Cond}(\mathrm{Ab})) \to D(\mathrm{Solid})$$

by $C \to C^{L\blacksquare} := C \otimes^L_{\mathbb{Z}} \mathbb{Z}_\blacksquare$.

(4') There is a unique symmetric monoidal structure $- \otimes^{L\blacksquare} - := - \otimes^L_{\mathbb{Z}_\blacksquare} -$ *on* $D(\mathrm{Solid})$ *such that the functor in (3') is symmetric monoidal. Moreover, it is left derived functor of (4)*[13].

We first analyze the structure of $\mathbb{Z}_\blacksquare[S]$.

Theorem 7.4. *For any profinite set* S*,* $C(S,\mathbb{Z})$ *is a free abelian group. Therefore,* $\mathbb{Z}_\blacksquare[S] \simeq \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z}) \simeq \prod_I \mathbb{Z}$ *for some set* I.

Proof. Let S injects into $\prod_\lambda\{0,1\}$ for some ordinal λ where the product is taking over ordinals $\mu < \lambda$. For each $\mu < \lambda$, by $S \to \{0,1\} \subset \mathbb{Z}$ projecting onto the μ-component, we get an idempotent element $e_\mu \in C(S,\mathbb{Z})$. Order the products $e_{\mu_1}\cdots e_{\mu_r}$ lexicographically and let E be the set of all such products that cannot be written as a linear combination of smaller such products. We will show that E forms a basis of $C(S,\mathbb{Z})$ by induction. If $\lambda = 0$, then we win. For limit ordinal λ, note that S is a filtered colimit of $S_\mu := \mathrm{im}(S \to \prod_{\mu'<\mu}\{0,1\})$. And for each S_μ there is a base E_μ. Since E is the union of all such E_μ, we win. Now we deal with successor ordinal. Assume $\lambda = \rho + 1$ and $T = S_\rho$. We have a closed immersion $S \hookrightarrow T \times \{0,1\}$. Let $S_i := S \cap (T \times \{i\})$. Note that S_i's are closed subsets of S with closed images covering T. Let T' be the intersection of $\mathrm{im}(S_1 \to T)$ and $\mathrm{im}(S_2 \to T)$. Hence we have a short exact sequence

$$0 \to C(T,\mathbb{Z}) \to C(S,\mathbb{Z}) \xrightarrow{\phi} C(T',\mathbb{Z}) \to 0$$

where $\phi(f) = f|_{T'\times\{0\}} - f|_{T'\times\{1\}}$. Then the products in E divides into the part starting with e_ρ and the part starting without e_ρ. The part started with e_ρ maps onto $C(T',\mathbb{Z})$ by sending $e_\rho e_{\mu_1}\cdots e_{\mu_r}$ to $e_{\mu_1}\cdots e_{\mu_r}$. Hence by induction we win. □

Remark 7.5. If $S = \lim_{\mathbb{N}} S_n$ is a countable limit of finite sets with surjective transition maps, then one can realize the above argument more explicit. Indeed, we can construct a sequence of compatible sections $c_n : S_n \to S$ inductively. Then we have

$$\mathbb{Z}_\blacksquare[S] = \lim_n \mathbb{Z}[S_n] \simeq \lim_n \bigoplus_{i=0}^n \mathbb{Z}[\pi_n(c_i(S_i \backslash \pi_i(c_{i-1}(S_{i-1}))))].$$

Hence $\mathbb{Z}_\blacksquare[S] \simeq \prod_{\mathbb{N}} \mathbb{Z}$ by the above limit.

Before we prove Theorem 7.1, we need some lemmas.

Lemma 7.6. *Let* S *be a profinite set and* $S_\bullet \to S$ *be a hypercover of profinite sets. Then the following complex is exact*

$$\cdots \to \mathbb{Z}_\blacksquare[S_1] \to \mathbb{Z}_\blacksquare[S_0] \to \mathbb{Z}_\blacksquare[S] \to 0$$

[13] As remarked before, the last statement is not true in general. We will prove it in Corollary 7.11

Proof. Since profinite sets have no higher cohomology, we know that

$$0 \to C(S,\mathbb{Z}) \to C(S_0,\mathbb{Z}) \to C(S_1,\mathbb{Z}) \to \cdots$$

is exact. By Theorem 7.4, everything is free, hence taking $\underline{\mathrm{Hom}}(-,\mathbb{Z})$ dual remains exact. □

Lemma 7.7. *We have* $\mathbb{Z}_\blacksquare[S] \in \mathrm{Solid}$ *and* $\mathbb{Z}_\blacksquare[S][0] \in D(\mathrm{Solid})$ *for any profinite set* S.

Proof. We need to show that for any profinite set T, the following map is an isomorphism

$$R\mathrm{Hom}(\mathbb{Z}_\blacksquare[T],\mathbb{Z}_\blacksquare[S]) \to R\mathrm{Hom}(\mathbb{Z}[T],\mathbb{Z}_\blacksquare[S]).$$

By Theorem 7.4, it suffices to show that

$$R\mathrm{Hom}(\mathbb{Z}_\blacksquare[T],\mathbb{Z}) \to R\mathrm{Hom}(\mathbb{Z}[T],\mathbb{Z})$$

is an isomorphism. By Theorem 7.4 again, we may assume $\mathbb{Z}_\blacksquare[T] \simeq \prod_J \mathbb{Z}$ for some set J. Observe that $R\mathrm{Hom}(\mathbb{Z}[T],\mathbb{Z}) = R\Gamma(T,\mathbb{Z}) = C(T,\mathbb{Z})$ by Theorem 4.6. By the short exact sequence

$$0 \to \prod_J \mathbb{Z} \to \prod_J \mathbb{R} \to \prod_J \mathbb{T} \to 0,$$

it is enough to show that $R\mathrm{Hom}(\prod_J \mathbb{R},\mathbb{Z}) = 0$ and $R\mathrm{Hom}(\prod_J \mathbb{T},\mathbb{Z}) = \bigoplus_J \mathbb{Z}[-1] \simeq C(T,\mathbb{Z})[-1]$. But this is done by Proposition 5.17. □

Remark 7.8. Lemma 7.7 holds for internal Hom, i.e.

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[T],\mathbb{Z}_\blacksquare[S]) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z}_\blacksquare[S]).$$

This is because Proposition 5.17 is state for internal Hom.

We now prove the main theorem that $\mathbb{Z}_\blacksquare$ is analytic.

Proof of Theorem 7.1. We want to show that for any profinite set S and $C_\bullet$ connected complex of the form $C_i = \bigoplus \prod \mathbb{Z}$, the following natural map is an isomorphism

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S],C_\bullet) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet).$$

Recall that we defined $\mathcal{M}(S,\mathbb{Z}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$ to be the "$\mathbb{Z}$-valued Haar measure". Similarly we define $\mathcal{M}(S,\mathbb{R}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{R})$ and $\mathcal{M}(S,\mathbb{T}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{T})$. Then the short exact sequence

$$0 \to \prod \mathbb{Z} \to \prod \mathbb{R} \to \prod \mathbb{T} \to 0$$

gives the short exact sequence of "measure spaces"

$$0 \to \mathcal{M}(S,\mathbb{Z}) \to \mathcal{M}(S,\mathbb{R}) \to \mathcal{M}(S,\mathbb{T}) \to 0.$$

Claim: We claim that it suffices to prove that

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),C_\bullet) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet)[-1]$$

is an isomorphism. Indeed, assume that we have the above isomorphism, then applying to $S = *$ we get

$$R\underline{\mathrm{Hom}}(\mathbb{T},C_\bullet) \simeq C_\bullet[-1] \simeq R\underline{\mathrm{Hom}}(\mathbb{Z}[1],C_\bullet).$$

Thus $R\underline{\mathrm{Hom}}(\mathbb{R},C_\bullet) \simeq 0$. Hence for general S we get

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{R}),C_\bullet) \simeq R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}(S,\mathbb{R}),R\underline{\mathrm{Hom}}(\mathbb{R},C_\bullet)) \simeq 0$$

Hence combining the claim again, we have

$$R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{Z}),C_\bullet)\simeq R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),C_\bullet)[1]\simeq R\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet)$$

as desired.

We will prove the claim by three steps.

Step 1: We prove for $C_\bullet = M[0]$ (where $M = \bigoplus\prod\mathbb{Z}$) concentrated at degree 0. Recall from Example 5.23 that compact abelian groups and $\mathbb{Z}[X]$ for $X \in$ CHaus are pseudocoherent. Hence viewing $\mathcal{M}(S,\mathbb{T})\simeq\prod\mathbb{T}$ as a compact abelian group, it is pseudocoherent. Therefore, $\mathcal{M}(S,\mathbb{T})\otimes^L \mathbb{Z}[T]$ is pseudocoherent for any profinite set T giving that $\mathcal{M}(S,\mathbb{T})$ is "internal pseudocoherent", i.e. $R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),-)$ commutes with filtered colimits. Hence, we can reduce to the case where $M=\mathbb{Z}$. But this is done by Lemma 7.7 (see also Remark 7.8).

Step 2: We prove for $C_\bullet$ bounded. This is by induction and the stupid truncations of $C_\bullet$.

Step 3: We prove for general $C_\bullet$. We will show that both $R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),C_\bullet)$ and $R\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet)[-1]$ are in cohomological degree ≤ 1 independent of S and $C_\bullet$. Then we have

$$R^n\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),C_\bullet)\simeq R^n\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),\sigma_{\leq n+2}C_\bullet)$$

and similarly for $R^n\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet)[-1]$. Hence we may reduce to the second step. Hence it suffices to bound the cohomological degree of $R\underline{\mathrm{Hom}}(\mathcal{M}(S,\mathbb{T}),C_\bullet)$ and $R\underline{\mathrm{Hom}}(\mathbb{Z}[S],C_\bullet)[-1]$. First observe that $C_\bullet$ naturally extends to a complex of $\mathbb{R}$-modules $C_{\bullet,\mathbb{R}}$. In fact, assuming $C_i = \bigoplus_{k\in K_i}\prod_{J_k}\mathbb{Z}$, we define

$$C_{i,\mathbb{R}} \coloneqq \bigoplus_{k\in K_i}\prod_{J_k}\mathbb{R}$$

itemwisely. To see that differential also extend naturally, it suffies to see the natural map

$$\mathrm{Hom}(C_i,C_{i+1,\mathbb{R}})\to\mathrm{Hom}(C_{i,\mathbb{R}},C_{i+1,\mathbb{R}})$$

is an isomorphism. Indeed, we may reduce to the case where $C_i=\prod\mathbb{Z}$. Hence it suffices to show that $R\mathrm{Hom}(\prod\mathbb{T},C_{i+1,\mathbb{R}})\simeq 0$. Again, since $\prod\mathbb{T}$ is pseudocoherent, it reduces to the case where $C_{i+1,\mathbb{R}}=\prod\mathbb{R}$ and then to the case where $C_{i+1,\mathbb{R}}=\mathbb{R}$. But the latter is done by Proposition 5.17 (2). Now let $C_{\bullet,\mathbb{T}}\coloneqq C_{\bullet,\mathbb{R}}/C_\bullet$, we have a short exact sequence of complexes

$$0\to C_\bullet\to C_{\bullet,\mathbb{R}}\to C_{\bullet,\mathbb{T}}\to 0.$$

Hence it suffices to prove the cohomological boundness for $C_{\bullet,\mathbb{R}}$ and $C_{\bullet,\mathbb{T}}$ instead of $C_\bullet$. Note that

$$C_{\bullet,\mathbb{R}}=\lim_i\tau_{\leq i}C_{\bullet,\mathbb{R}} \text{ and } C_{\bullet,\mathbb{T}}=\lim_i\tau_{\leq i}C_{\bullet,\mathbb{T}}$$

it suffices to prove the cohomological boundness for bounded complexes $\tau_{\leq i}C_{\bullet,\mathbb{R}}$ and $\tau_{\leq i}C_{\bullet,\mathbb{T}}$. Hence the problem reduces to termwise again. Since $\mathcal{M}(S,\mathbb{T})$ and $\mathbb{Z}[S]$ are pseudocoherent, we may assume that $C_{i,\mathbb{R}}=\prod\mathbb{R}$ and $C_{i,\mathbb{T}}=\prod\mathbb{T}$.

We first treat the case of $C_{i,\mathbb{T}}$ and $\ker(d_{i,\mathbb{T}})$. Note that for $C_{i,\mathbb{T}}$ it follows from the computation of Proposition 5.17. As for $\ker(d_{i,\mathbb{T}})$, observe that as the kernel compact abelian groups, it is a compact abelian group. Hence applying Pontryagin duality $\mathbb{D}(-)$, we get a discrete abelian group $\mathbb{D}(\ker(d_{i,\mathbb{T}}))$. Now using a two term resolution of $\mathbb{D}(\ker(d_{i,\mathbb{T}}))$ (since $\mathbb{Z}$ is a PID) and applying

Pontryagin duality again, we get a short exact sequence
$$0 \to \ker(d_{i,\mathbb{T}}) \to \prod_{J_1} \mathbb{T} \to \prod_{J_2} \mathbb{T} \to 0.$$
Then we win by Proposition 5.17.

We then treat the case of $C_{i,\mathbb{R}}$ and $\ker(d_{i,\mathbb{R}})$. Again $C_{i,\mathbb{R}}$ follows from the computation of Proposition 5.17. Note that $\ker(d_{i,\mathbb{R}})$ is the kernel of $\prod_I \mathbb{R} \to \prod_J \mathbb{R}$ which is the $\mathbb{R}$-linear extension of the corresponding map $\prod_I \mathbb{Z} \to \prod_J \mathbb{Z}$. The latter is the dual of $g : \bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$. Hence $d_{i,\mathbb{R}}$ is the dual of
$$g_{\mathbb{R}} : \bigoplus_J \mathbb{R} \to \bigoplus_I \mathbb{R}$$
which can be decompose into a split surjection and a split injection. Hence $\operatorname{coker}(g_{\mathbb{R}}) \simeq \bigoplus_K \mathbb{R}$ is a $\mathbb{R}$-vector space. Thus, $\ker(d_{i,\mathbb{R}}) \simeq \operatorname{Hom}_{\mathbb{R}}(\operatorname{coker}(g_{\mathbb{R}}), \mathbb{R}) \simeq \prod_K \mathbb{R}$. Then we win. □

Example 7.9. We give two immediate examples of solid abelian groups.

(1) Any discrete abelian group M is a solid abelian group. In fact, one can always resolve M by direct sums of $\mathbb{Z}$, i.e. $\bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to M \to 0$.
(2) Any limit or colimit of discrete abelian groups is a solid abelian group. In particular, the p-adic integers $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$ is solid.

We now show some corollaries of Theorem 7.1.

Corollary 7.10. *The category* Solid *is generated by compact projective generators* $\prod_I \mathbb{Z}$. *Moreover, the derived enhancement* $D(\text{Solid})$ *is also compactly generated by bounded complexes* $C_\bullet$ *where* C_i *is of the form* $\prod_I \mathbb{Z}$. *The full subcategory of compact objects* $D(\text{Solid})^\omega$ *is equivalent to* $D^b(\mathbb{Z})^{\text{op}}$ *by the functor*
$$D(\text{Solid})^\omega \to D^b(\mathbb{Z})^{\text{op}}$$
sending $C_\bullet$ *to* $R\text{Hom}(C_\bullet, \mathbb{Z})$.

Proof. Except for the last statement, all other statements are just explicit description of Proposition 7.3. As for the last statement, note that boundedness comes from the identification
$$R\text{Hom}(\mathbb{Z}_\blacksquare[S], \mathbb{Z}) \simeq R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) \simeq R\Gamma(S, \mathbb{Z}) = C(S, \mathbb{Z}).$$
Then one easily finds an inverse $D^b(\mathbb{Z})^{\text{op}} \to D(\text{Solid})^\omega$ by sending $C_\bullet$ to $R\underline{\text{Hom}}(C_\bullet, \mathbb{Z})$. □

Corollary 7.11. *We have the following:*

(1) The derived solidification $\mathbb{R}^{L\blacksquare}$ *of* $\mathbb{R}$ *is 0.*
(2) For any profinite set S, *the derived solidification* $\mathbb{Z}[S]^{L\blacksquare}$ *is* $\mathbb{Z}_\blacksquare[S]$. *Hence* $\mathbb{Z}_\blacksquare[S] \otimes^{L\blacksquare} \mathbb{Z}_\blacksquare[T] \simeq \mathbb{Z}_\blacksquare[S \times T]$. *Moreover, for any products* $\prod_I \mathbb{Z}$ *and* $\prod_J \mathbb{Z}$, *we have*
$$\prod_I \mathbb{Z} \otimes^{L\blacksquare} \prod_J \mathbb{Z} \simeq \prod_{I \times J} \mathbb{Z}.$$
(3) The symmetric monoidal structure $- \otimes^{L\blacksquare} -$ *on* $D(\text{Solid})$ *is the left derived of* $- \otimes^\blacksquare -$ *on* Solid.

Proof. For (1), notice that the claim in the proof of Theorem 7.1 shows that $R\mathrm{Hom}(\mathbb{R}, C_\bullet) \simeq 0$ for any connected complex $C_\bullet$ whose terms are of the form $\bigoplus \prod \mathbb{Z}$. For general $C_\bullet \in D(\mathrm{Solid})$, we first use Postnikov limit to reduce to the case where $C_\bullet$ is connected. Then we use projective resolution to reduce to the previous case.

As for (2), the first statement is by the left adjointness of derived solidification and the definition of being solid. The second statement follows from derived solidification being symmetric monoidal, i.e. $(C \otimes^L D)^{L\blacksquare} = C^{L\blacksquare} \otimes^{L\blacksquare} D^{L\blacksquare}$. To show the last statement, note that we can write M and N as $\underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z}, \mathbb{Z})$ and $\underline{\mathrm{Hom}}(\bigoplus_J \mathbb{Z}, \mathbb{Z})$. Since one can always realize $\bigoplus_I \mathbb{Z}$ (resp. $\bigoplus_J \mathbb{Z}$) as the retraction of $C(S, \mathbb{Z})$ for some profinite set S (resp. $C(T, \mathbb{Z})$ for some profinite set T), we can reduce to the case where $M = \mathbb{Z}_\blacksquare[S]$ and $N = \mathbb{Z}_\blacksquare[T]$. Hence we win.

To show (3), it suffices to show that for compact projective $M, N \in \mathrm{Solid}$, we have $M \otimes^{L\blacksquare} N = M \otimes^{\blacksquare} N$. Assuming $M = \mathbb{Z}_\blacksquare[S]$ and $N = \mathbb{Z}_\blacksquare[T]$, this is done by (2). □

We now provides a few examples connecting solid abelian groups and some classic objects. The first example concerns singular homology of a CW complex. The following discusses power series and p-adic integers.

Example 7.12. Let X be a CW complex and $H_\bullet(X, \mathbb{Z})$ be its singular homology complex. Identifying $D(\mathrm{Solid})$ with $\mathrm{Ind}(D^b(\mathbb{Z})^{op})$ as in Corollary 7.10, we have

$$\mathbb{Z}[X]^{L\blacksquare} \simeq H_\bullet(X, \mathbb{Z}).$$

To see this, first observe that both sides commutes with filtered colimits, so one can reduce to the case where X is a finite CW complex, in particular a compact Hausdorff space. Hence resolving X by a hypercover of extremally disconnected sets, we know that $\mathbb{Z}[X]^{L\blacksquare}$ is reflexive. Thus, it suffices to show the dual statement, i.e.

$$R\mathrm{Hom}(\mathbb{Z}[X]^{L\blacksquare}, \mathbb{Z}) \simeq R\mathrm{Hom}(\mathbb{Z}[X], \mathbb{Z}) \simeq R\Gamma_{\mathrm{sing}}(X, \mathbb{Z})$$

which is done by Theorem 4.6 combining the fact that X is a CW complex.

Example 7.13. Using the null-sequence as in Remark 7.2, one can recover the p-adic integers. Let Q be the quotient of $\mathrm{NS}(\mathbb{Z})$ via "multiplication by $T - p$". Again, for any solid abelian group M, we view a map $\mathrm{NS}(\mathbb{Z}) \to M$ as a convergent sequence $m_0, m_1, \ldots$ in M. Then "multiplication by $T - p$" is given by sending $m_k \mapsto p^k m_k$. Taking derived solidification yields

$$0 \to \mathrm{NS}_\blacksquare(\mathbb{Z}) \simeq \mathbb{Z}[[T]] \xrightarrow{T-p} \mathrm{NS}_\blacksquare(\mathbb{Z}) \simeq \mathbb{Z}[[T]] \to \mathbb{Z}_p \to 0.$$

Example 7.14. Identify $\mathbb{Z}[[T_i]]$ with $\prod_{n\in\mathbb{N}} \mathbb{Z}T_i^n$ for $i = 1, 2$, we have

$$\mathbb{Z}[[T_1]] \otimes^{L\blacksquare} \mathbb{Z}[[T_2]] \simeq \mathbb{Z}[[T_1, T_2]]$$

by Corollary 7.11 (2). Hence by Example 7.13 above, we have

$$\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}[[T]] \simeq \mathbb{Z}_p[[T]].$$

Moreover, since $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_l$ can be identified with

$$\mathbb{Z}_p[[T]] \xrightarrow{T-l} \mathbb{Z}_p[[T]],$$

we have $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_l \simeq \mathbb{Z}_p$ if $p = l$ and $\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_l \simeq 0$ if $p \neq l$ as $T - l$ is invertible.

With the above interpretation of $\mathbb{Z}_p$, we prove the following fact.

Proposition 7.15. *The pre-analytic ring $\mathbb{Z}_{p,\blacksquare}$ as in Example 6.4 is analytic.*

Proof. By writing $\mathbb{Z}_p$ as the quotient of $\mathbb{Z}[[T]]$ via multiplication by $T-p$, we can represent $\mathbb{Z}_\blacksquare[S]\otimes^{L\blacksquare} \mathbb{Z}_p$ by the two term complex $\mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \mathbb{Z}[[T]] \xrightarrow{T-p} \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \mathbb{Z}[[T]]$. Using Corollary 7.11 (2), we can identify the above as $\prod_I \mathbb{Z}[[T]] \xrightarrow{T-p} \prod_I \mathbb{Z}[[T]]$ which is $\prod_I \mathbb{Z}_p$ by (AB4*): taking product is exact. Hence we have

$$\mathbb{Z}_\blacksquare[S] \otimes^{L\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_{p,\blacksquare}[S].$$

Let C be a connected complex of solid abelian groups whose terms are direct sums of $\mathbb{Z}_{p,\blacksquare}[S]$ for varying extremally disconnected S. Note that C is also solid. We then have

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathbb{Z}_p}(\mathbb{Z}_{p,\blacksquare}[S], C) &\simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}_p}(\mathbb{Z}_\blacksquare[S] \otimes^{L\blacksquare} \mathbb{Z}_p, C) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}_\blacksquare[S], C) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], C) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}_p}(\mathbb{Z}_p[S], C) \end{aligned}$$

as desired. ☐

Remark 7.16 (Warning). We warn the reader here that using product of $\mathbb{Z}$ to resolve $\mathbb{Z}_p$ here is crucial in Proposition 7.15. One may be tempted to resolve a finitely generated $\mathbb{Z}$-algebra A by direct sums and mimic the proof to show $A_\blacksquare$ is analytic. But this will not go through since $(\prod \mathbb{Z}) \otimes (\bigoplus \mathbb{Z}) \neq \prod \bigoplus \mathbb{Z}$ in general. Hence to prove $A_\blacksquare$ is analytic for a finitely generated $\mathbb{Z}$-algebra A, one cannot trivially resolve A by direct sums of $\mathbb{Z}$. Therefore, some work needs to be done. However, it is a lot more easier to prove for $(A, \mathbb{Z})_\blacksquare$ (without any finite assumptions). These will come up in the next section.

Note that all the "good behaviour" under solidification above are nonarchimedean by nature, e.g. $\mathbb{Z}_p$. And archimedean examples like $\mathbb{R}$ behave poorly. This means that the notion of solid needs to be modified in the archimedean world which is called liquid as in [SCc]. We end this section by giving the following example.

Example 7.17. We define $R_{l^1} \coloneqq (\mathbb{R}, R_{l^1}[-], \alpha)$ where

$$R_{l^1}[S] \coloneqq \mathcal{M}^b(S, \mathbb{R}) \simeq \bigcup_{r>0} \lim_i \mathbb{R}[S_i]_{l^1 \leq r}$$

where $\mathcal{M}^b(S, \mathbb{R})$ is the bounded measure on S and $\mathbb{R}[S_i]_{l^1 \leq r}$ is the subspace of l^1-norm at most r. In this definition, R_{l^1} is not an analytic ring due to the Ribe extension

$$R^1\mathrm{Hom}_{\mathbb{R}}(R_{l^1}[S], R_{l^1}[S]) \neq 0$$

is nontrivial.

8. Analytic Structures

We will discuss two different analytic structures $(A,\mathbb{Z})_\blacksquare$ and $A_\blacksquare$. Along the way the mechinary of six-functors will be developed. Two main examples are $(\mathbb{Z}[T],\mathbb{Z})_\blacksquare$ and $\mathbb{Z}[T]_\blacksquare$[14], where the former can be viewed as the "canonical compactification" of the latter. We first recall the following two definitions from Example 6.4:

(1) For a discrete ring A, we can associate it with a pre-analytic ring $(A,\mathbb{Z})_\blacksquare = (A,(A,\mathbb{Z})_\blacksquare[-],\alpha)$, where $(A,\mathbb{Z})_\blacksquare[S] \coloneqq \mathbb{Z}_\blacksquare[S]\otimes_\mathbb{Z} A$ and α is given by the one on $\mathbb{Z}_\blacksquare$.

(2) For a finitely generated $\mathbb{Z}$-algebra A, we can form a pre-analytic ring $A_\blacksquare = (A,A)_\blacksquare = (A,A_\blacksquare[-],\alpha)$, where $A_\blacksquare[S] = \lim_i A[S_i]$ and α similarly as above.

One needs some work to prove that $A_\blacksquare$ is analytic. But proving the analyticity of $(A,\mathbb{Z})_\blacksquare$ is easy:

Lemma 8.1. *For a discrete ring A, the pre-analytic ring $(A,\mathbb{Z})_\blacksquare$ is analytic.*

Proof. We should check that for any

$$C_\bullet = \cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0$$

with $C_i = (\bigoplus_{j\in J_i}\prod_{k\in K_j}\mathbb{Z})\otimes A \simeq \bigoplus_{j\in J_i}(\prod_{k\in K_j}\mathbb{Z}\otimes A)$, the following map is an isomorphism

$$R\underline{\mathrm{Hom}}_A((A,\mathbb{Z})_\blacksquare[S],C_\bullet) \to R\underline{\mathrm{Hom}}_A(A[S],C_\bullet).$$

Note that are isomorphisms

$$(A,\mathbb{Z})_\blacksquare[S] \coloneqq \mathbb{Z}_\blacksquare[S]\otimes A \xrightarrow{\simeq} \mathbb{Z}_\blacksquare[S]\otimes^\blacksquare A \xrightarrow{\simeq} \mathbb{Z}_\blacksquare[S]\otimes^{L\blacksquare} A.$$

The second isomorphism is by Corollary 7.10 that $\mathbb{Z}_\blacksquare[S]$ is projective. To see the first isomorphism, note that A is discrete, hence resolving A by direct sums of $\mathbb{Z}$, it suffices to show the case were $A = \bigoplus \mathbb{Z}$, which is obvious. Note that $C_\bullet$ is also solid. Thus we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_A((A,\mathbb{Z})_\blacksquare,C_\bullet) &\simeq R\underline{\mathrm{Hom}}_A(\mathbb{Z}_\blacksquare[S]\otimes^{L\blacksquare} A,C_\bullet) \\ &\simeq R\underline{\mathrm{Hom}}_\mathbb{Z}(\mathbb{Z}_\blacksquare[S],C_\bullet) \\ &\simeq R\underline{\mathrm{Hom}}_\mathbb{Z}(\mathbb{Z}[S],C_\bullet) \\ &\simeq R\underline{\mathrm{Hom}}_A(A[S],C_\bullet). \end{aligned}$$

□

Remark 8.2 (Definition). The proof of Lemma 8.1 suggests that the definition of $(A,\mathbb{Z})_\blacksquare$ should include a larger class of rings, not just the discrete ones. Indeed, for any condensed ring A whose underlying condensed abelian group is solid, we can define the pre-analytic ring $(A,\mathbb{Z})_\blacksquare$ as

$$(A,\mathbb{Z})_\blacksquare[S] \coloneqq \mathbb{Z}_\blacksquare[S]\otimes^\blacksquare A$$

where S is profinite. Moreover, the proof of Lemma 8.1 formally implies that $(A,\mathbb{Z})_\blacksquare$ is analytic.

This uniforms a few examples that we have seen above.

(1) When A is discrete, $(A,\mathbb{Z})_\blacksquare$ specializes to the definition in Example 6.4.

[14]Throughout this section $\mathbb{Z}[T]$ is the polynomial algebra instead of the free condensed abelian group associated to some condensed set T. And $\mathbb{Z}[[T]]$ denotes the formal power series with one variable.

(2) Another example would be $A = \mathbb{Z}[[T]] \simeq \prod \mathbb{Z}$. Then Corollary 7.11 (2) implies that for any profinite set S

$$\mathbb{Z}[[T]]_\blacksquare[S] \simeq \prod \mathbb{Z}[[T]] \simeq \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \mathbb{Z}[[T]] = (\mathbb{Z}[[T]], \mathbb{Z})_\blacksquare[S].$$

Hence we have $\mathbb{Z}[[T]]_\blacksquare \simeq (\mathbb{Z}[[T]], \mathbb{Z})_\blacksquare$.

(3) The proof of Proposition 7.15 also shows that $\mathbb{Z}_{p,\blacksquare} \simeq (\mathbb{Z}_p, \mathbb{Z})_\blacksquare$. This is mainly because we can resolve $\mathbb{Z}_p$ by $\mathbb{Z}[[T]]$.

The main goal of this section is to prove that $A_\blacksquare$ is analytic and develop the lower shriek functor. We will start with the case where $A = \mathbb{Z}[T]$ is the polynomial algebra and deduce the general case from it.

Proposition 8.3. *The pre-analytic ring $\mathbb{Z}[T]_\blacksquare$ is analytic, i.e. for all connective complexes C whose terms are of the form $\bigoplus_i \mathbb{Z}[T]_\blacksquare[S_i]$ with S_i extremally disconnected,*

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T]_\blacksquare[S], C) \xrightarrow{\simeq} R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T][S], C)$$

is an isomorphism.

We start with constructing the "functions near the boundary of $\mathrm{Spec}(\mathbb{Z}[T])$".

Construction 8.4. We call $\mathbb{Z}((T^{-1}))$, endowed with natural T^{-1}-adic topology, "functions near the boundary of $\mathrm{Spec}(\mathbb{Z}[T])$". As a condensed abelian group, it is a colimit of $\mathbb{Z}[[T^{-1}]] \simeq \prod_{\mathbb{N}} \mathbb{Z}$, i.e.

$$\mathrm{colim}(\mathbb{Z}[[T^{-1}]] \xrightarrow{T^{-1}} \mathbb{Z}[[T^{-1}]] \xrightarrow{T^{-1}} \cdots).$$

Since $\mathbb{Z}((T^{-1}))$ is a filtered colimit of solid modules, itself is solid. Moreover, it is a natural $\mathbb{Z}[T]$-algebra. Hence we have $\mathbb{Z}((T^{-1})) \in D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$.

We first establish a few lemmas concerning properties of $\mathbb{Z}((T^{-1}))$.

Lemma 8.5. *The objest $\mathbb{Z}((T^{-1})) \in D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$ is compact, i.e. $R\underline{\mathrm{Hom}}(\mathbb{Z}((T^{-1})), -)$ commutes with filtered colimits.*

Proof. Note that $\mathrm{NS}_\blacksquare(\mathbb{Z}) = \mathbb{Z}[[T]] \simeq \prod_{\mathbb{N}} \mathbb{Z}$ is a compact solid module by Remark 7.2 and Proposition 7.3. Then Theorem 6.12 shows that $\mathbb{Z}[[U]] \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is compact in $D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$. Now it follows from the resolution

$$0 \to \mathbb{Z}[[U]] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \xrightarrow{UT-1} \mathbb{Z}[[U]] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \to 0.$$

□

Lemma 8.6. *The ring $\mathbb{Z}((T^{-1})) \in D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$ is idempotent, i.e. the natural map*

$$\mathbb{Z}((T^{-1})) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1})) \to \mathbb{Z}((T^{-1}))$$

is an isomorphism. In particular, $\mathbb{Z}((T^{-1}))$-modules form a full subcategory in $D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$, i.e. $M \in D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$ admits a (automatically unique) $\mathbb{Z}((T^{-1}))$-module structure if and only if the map

$$M \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1})) \to M$$

is an isomorphism.

Proof. Again we use the resolution as in Lemma 8.5. Indeed we have

$$\begin{aligned}\mathbb{Z}((T^{-1}))\otimes^{L}_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}\mathbb{Z}((T^{-1})) &\simeq \operatorname{cofib}(\mathbb{Z}[[U]]\otimes_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}\mathbb{Z}((T^{-1}))\xrightarrow{UT-1}\mathbb{Z}[[U]]\otimes_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare}\mathbb{Z}((T^{-1})))\\ &\simeq \operatorname{cofib}(\mathbb{Z}[[U,T^{-1}]][T]\xrightarrow{UT-1}\mathbb{Z}[[U,T^{-1}]][T])\\ &\simeq \mathbb{Z}((T^{-1}))\end{aligned}$$

where the second isomorphism is by Corollary 7.11 (2) and the fact that analytic structure of $(\mathbb{Z}[T],\mathbb{Z})_\blacksquare$ is inherited from $\mathbb{Z}_\blacksquare$. The last assertion follows from general nonsense of idempotent monad, c.f. [nA, idempotent monad]. □

Lemma 8.7. *For any complex $C\in D(\mathrm{Cond}(\mathrm{Ab}))$ whose terms are direct sums of products of $\mathbb{Z}[T]$, it is $\mathbb{Z}((T^{-1}))$-orthogonal, i.e. we have*

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),C)\simeq 0.$$

Moreover, for any $\mathbb{Z}((T^{-1}))$-module M, we have $R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(M,C)\simeq 0$.

Proof. By writing C as the limit of its stupid truncations, we may assume that C is connective. Since $\mathbb{Z}((T^{-1}))$ is compact, by writing C as the colimit of its stupid truncations[15], we may reduce to the case where $C=\bigoplus\prod\mathbb{Z}[T]$. Again, since $\mathbb{Z}((T^{-1}))$ is compact, we may assume that $C=\prod\mathbb{Z}[T]$. Finally, we reduce to the case that $C=\mathbb{Z}[T]$. Using the same resolution as above, we can compute $R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),C)$ as

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),\mathbb{Z}[T]) &\simeq \operatorname{fib}(R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[U]]\otimes_{\mathbb{Z}}\mathbb{Z}[T],\mathbb{Z}[T])\xrightarrow{UT-1}R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[U]]\otimes_{\mathbb{Z}}\mathbb{Z}[T],\mathbb{Z}[T]))\\ &\simeq \operatorname{fib}(R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[U]],\mathbb{Z}[T])\xrightarrow{UT-1}R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[U]],\mathbb{Z}[T]))\\ &\simeq \operatorname{fib}(R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[U]],\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T]\xrightarrow{UT-1}R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[U]],\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T])\\ &\simeq \operatorname{fib}((\mathbb{Z}[U^{-1}]/\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T]\xrightarrow{UT-1}(\mathbb{Z}[U^{-1}]/\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T])\\ &\simeq \operatorname{fib}(\mathbb{Z}[U^{-1},T]/\mathbb{Z}[T]\xrightarrow{UT-1}\mathbb{Z}[U^{-1},T]/\mathbb{Z}[T])\\ &\simeq 0\end{aligned}$$

The only nontrivial parts are the third and forth isomorphism. The third one is by the fact that $\mathbb{Z}[[U]]$ is compact and $\mathbb{Z}[T]$ is a direct sum of $\mathbb{Z}$. To see the forth isomorphism, we define the residue pairing[16]

$$\mathbb{Z}[[U]]\times\mathbb{Z}[U^{-1}]/\mathbb{Z}\xrightarrow{\text{can}}\mathbb{Z}((U))/\mathbb{Z}[[U]]\xrightarrow{\text{Res}}\mathbb{Z}$$

where the first map is the multiplication map and the second map is the residue map, i.e. take the coefficient of U^{-1}. Clearly this is a perfect pairing. Moreover, the pairing is compatible with multiplication by U since $\mathrm{Res}(f(g\cdot U))=\mathrm{Res}((f\cdot U)g)$ where $f\in\mathbb{Z}[[U]]$ and $g\in\mathbb{Z}[U^{-1}]/\mathbb{Z}$. Hence the identification $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[U]],\mathbb{Z})\simeq\mathbb{Z}[U^{-1}]/\mathbb{Z}$ is compatible with multiplication by U.

[15] Note the difference between realizing C as the limit of its canonical filtration (c.f. [Sta, 0118]) as in proving Theorem 7.1.

[16] One can compare this with the residues of differentials on curves, c.f. [Har77, Remark 7.14].

The last statment follows immediately from adjunction and identifying M with $M \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1}))$ by Lemma 8.6. $\square$

Lemma 8.8. *For any profinite set S, the cofibre*

$$\operatorname{cofib}(\mathbb{Z}_\blacksquare[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \mathbb{Z}[T]_\blacksquare[S])$$

is a natural $\mathbb{Z}((T^{-1}))$-module.

Proof. Identify $\mathbb{Z}_\blacksquare[S]$ and $\mathbb{Z}[T]_\blacksquare[S]$ with $\prod_I \mathbb{Z}$ and $\prod_I \mathbb{Z}[T]$ respectively. It suffices to show that there is an isomorphism

$$\operatorname{cofib}((\prod_I \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \prod_I \mathbb{Z}[T]) \xrightarrow{\simeq} \operatorname{cofib}((\prod_I \mathbb{Z}[[T^{-1}]]) \otimes_{\mathbb{Z}[[T^{-1}]]} \mathbb{Z}((T^{-1})) \to \prod_I \mathbb{Z}((T^{-1}))).$$

Denote the former as C and the latter as C'. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\prod_I \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[T] & \longrightarrow & \prod_I \mathbb{Z}[T] & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & (\prod_I \mathbb{Z}[[T^{-1}]]) \otimes_{\mathbb{Z}[[T^{-1}]]} \mathbb{Z}((T^{-1})) & \longrightarrow & \prod_I \mathbb{Z}((T^{-1})) & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

By snake lemma, we have an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Note that clearly $\ker(g) \simeq 0$. Hence it suffices to show that $\operatorname{coker}(f) \to \operatorname{coker}(g)$ is an isomorphism. However, since $\mathbb{Z}((T^{-1}))/\mathbb{Z}[T] \simeq T^{-1}\mathbb{Z}[[T^{-1}]]$, we can compute $\operatorname{coker}(f) \to \operatorname{coker}(g)$ as

$$T^{-1} \prod_I \mathbb{Z}[[T^{-1}]] \to \prod_I T^{-1}\mathbb{Z}[[T^{-1}]]$$

which is clearly an isomorphism. $\square$

We now prove Proposition 8.3.

Proof of Proposition 8.3. We need to show that for all connective complexes C whose terms are of the form $\bigoplus \prod \mathbb{Z}[T]$,

$$R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T]_\blacksquare[S], C) \xrightarrow{\simeq} R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T][S], C)$$

is an isomorphism. By Lemma 8.1, we know that

$$R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}((\mathbb{Z}[T],\mathbb{Z})_\blacksquare[S], C) \xrightarrow{\simeq} R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T][S], C).$$

Hence it suffices to show

$$R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T]_\blacksquare[S], C) \to R\underline{\operatorname{Hom}}_{\mathbb{Z}[T]}((\mathbb{Z}[T],\mathbb{Z})_\blacksquare[S], C)$$

is an isomorphism. However, after identifying $(\mathbb{Z}[T],\mathbb{Z})_\blacksquare[S]$ with $\mathbb{Z}_\blacksquare[S] \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, we win by Lemma 8.8 and Lemma 8.7. $\square$

Remark 8.9. We give an intuitive interpretation of what happens above. Geometrically, we can think $|\mathbb{P}^{1,\mathrm{ad}}|$ as the disjoint union of $|\mathbb{A}^{1,\mathrm{ad}}|$ and $|\{\infty\}|$. Then identify $D(\mathbb{Z}[T]_\blacksquare)$ as functions on $\mathbb{A}^{1,\mathrm{ad}}$ and $D((\mathbb{Z}[[T^{-1}]],\mathbb{Z})_\blacksquare)$ as functions on $\mathbb{P}^{1,\mathrm{ad}}$ supported at $\{\infty\}$. Even though Lemma 8.7 and Lemma 8.8 do not give the localization sequence (in the sense of [Sta, 05RA])

$$D((\mathbb{Z}[[T^{-1}]],\mathbb{Z})_\blacksquare) \to D(\mathbb{P}^{1,\mathrm{ad}}) \to D(\mathbb{Z}[T]_\blacksquare)$$

directly, they give the localization sequence after localizing at $\mathbb{A}^{1,\mathrm{ad}}$:

$$D((\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare) \to D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare) \to D(\mathbb{Z}[T]_\blacksquare).$$

Hence we can think $(\mathbb{Z}[T],\mathbb{Z})_\blacksquare$ as functions on the "canonical compactification" of $\mathbb{A}^{1,\mathrm{ad}}$. This also justifies $(\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare$ as "functions near the boundary of $\mathbb{A}^{1,\mathrm{ad}}$".

Following the ideas of the remark above, we formalize the following proposition. In particular, we will show the lower shriek functor in this context.

Theorem 8.10. *Consider two natural maps of analytic rings $i : (\mathbb{Z}[T],\mathbb{Z})_\blacksquare \to (\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare$ and $j : (\mathbb{Z}[T],\mathbb{Z})_\blacksquare \to \mathbb{Z}[T]_\blacksquare$. We have the following localization sequence*

$$D((\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare) \xrightarrow{i_*} D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare) \xrightarrow{j^*} D(\mathbb{Z}[T]_\blacksquare)$$

where i_* is the forgetful functor and j^* is the base change functor. Moreover,

(1) i_* has a left adjoint i^* and a right adjoint $i^!$. The left adjoint i^* is given by

$$i^*(C) = C \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1}))$$

and the right adjoint $i^!$ is given by

$$i^!(C) = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),C)$$

for $C \in D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare)$.

(2) j^* has a fully faithful left adjoint $j_!$ and a fully faithful right adjoint j_* which is the forgetful functor. The left adjoint $j_!$ is given by

$$j_!j^*(C) = \mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} C$$

and the right adjoint j_* is given by

$$j_*j^*(C) = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))),C)$$

for $C \in D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare)$.

(3) There are excision triangles

$$j_!j^*(C) \to C \to i_*i^*(C)$$

and

$$i_*i^!(C) \to C \to j_*j^*(C)$$

for $C \in D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare)$.

Proof. First note that by Proposition 8.3 and its proof, $j : (\mathbb{Z}[T], \mathbb{Z})_\blacksquare \to \mathbb{Z}[T]_\blacksquare$ is a well-defined map of analytic rings. By Proposition 6.23, j^* is the left adjoint of the fully faithful forgetful functor j_*.

We then show (1). By Lemma 8.6, we know that $D((\mathbb{Z}((T^{-1})), \mathbb{Z})_\blacksquare) \xrightarrow{i_*} D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$ is a fully faithful embedding. By the definition of $D((\mathbb{Z}((T^{-1})), \mathbb{Z})_\blacksquare)$, it is stable under limits and colimits. Hence i_* admits a left adjoint i^* and a right adjoint $i^!$. It is easy to check the formulas for i^* and $i^!$ just as in the classical algebraic geometry.

Claim: j^* realizes $D(\mathbb{Z}[T]_\blacksquare)$ as the Verdier quotient of $D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$ by $D((\mathbb{Z}((T^{-1})), \mathbb{Z})_\blacksquare)$. In fact, if $C \in D((\mathbb{Z}((T^{-1})), \mathbb{Z})_\blacksquare)$, then for any $C' \in D(\mathbb{Z}[T]_\blacksquare)$ we have

$$R\underline{\mathrm{Hom}}(j^*(C), C') \simeq R\underline{\mathrm{Hom}}(C, j_*(C')) \simeq 0$$

by Lemma 8.7 and the fact that $\mathbb{Z}((T^{-1}))$ is compact. Observe also that the kernel of j^* is generated by objects of the form

$$\mathrm{fib}(C \to j_* j^*(C))$$

for some $C = (\mathbb{Z}[T], \mathbb{Z})_\blacksquare[S]$, c.f. [Sta, Lemma 05RK]. Then Lemma 8.8 shows that they are $\mathbb{Z}((T^{-1}))$-modules. Now we show (2). Consider the endofunctor

$$D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare) \to D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$$

given by $C \mapsto \mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} C$. We claim that this is the composition of j^* and $j_!$ where the latter the fully faithful left adjoint of j^*. Note that once the left adjointness is proved, then we have $j^* j_!$ being the left adjoint of $j^* j_*$ which is identity due to j_* being fully faithful. Hence $j^* j_!$ is identity showing $j_!$ being fully faithful. To see the factorization, note that since $\mathbb{Z}((T^{-1}))$ is idempotent by Lemma 8.6, we have

$$\mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} C \simeq \mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1})) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} C \simeq 0$$

for any $C \in D((\mathbb{Z}((T^{-1})), \mathbb{Z})_\blacksquare)$. Then by the unviersal property of the Verdier quotient, we have the desired factorization. Since j^* is essential surjective, now it suffices to show that

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(j_! j^*(C), C') \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(C, j_* j^*(C')) \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]_\blacksquare}(j^*(C), j^*(C'))$$

for any $C, C' \in D((\mathbb{Z}[T], \mathbb{Z})_\blacksquare)$. Note that $\mathrm{fib}(C' \to j_* j^*(C'))$ is a $\mathbb{Z}((T^{-1}))$-module by the claim. Note also $j_! j^*(C) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1})) \simeq 0$ by the definition of $j_! j^*$ and $\mathbb{Z}((T^{-1}))$ being idempotent. Hence we have

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(j_! j^*(C), C') \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(j_! j^*(C), j_* j^*(C'))$$

By definition $\mathrm{cofib}(j_! j^*(C) \to C)$ is an $\mathbb{Z}((T^{-1}))$-module, then we have

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(j_! j^*(C), j_* j^*(C')) \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(C, j_* j^*(C'))$$

by Lemma 8.7. This completes the proof of the claimed equivalence. For the formula in $j_* j^*(C)$ note that

$$j_* j^*(C) \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T], j_* j^*(C)) \simeq R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(j_! j^*(\mathbb{Z}[T]), C) = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))), C).$$

One checks (3) easily from the above.

□

Remark 8.11. Once we realize $D(\mathbb{Z}[T]_\blacksquare)$ as the Verdier quotient of $D((\mathbb{Z}[T],\mathbb{Z})_\blacksquare)$ by $D((\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare)$, the proof of Proposition 8.10 then can be done by the general nonsense described in [SCb, Construction 5.2] and [Kra08, Theorem 5.6.1, Example 5.9]. Indeed, the key ingredient is that $\mathbb{Z}((T^{-1}))$ is compact and idempotent. Which implies that $D((\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare)$ is closed under limits and colimits and therefore, j^* commutes with limits and colimits. Hence we have the left adjoint $j_!$ and the right adjoint j_*.

The following remark is taken from [RC, Lecture 9].

Remark 8.12. The localization sequence

$$D((\mathbb{Z}[[T^{-1}]],\mathbb{Z})_\blacksquare) \to D(\mathbb{P}^{1,\mathrm{ad}}) \to D(\mathbb{Z}[T]_\blacksquare)$$

described previously is well studied in terms of adic spaces after pulling back to $\mathbb{Q}_p$. In fact, we get the following localization sequence after applying $-\otimes^{L\blacksquare}\mathbb{Q}_p$

$$D((\mathbb{Z}_p[[T^{-1}]][1/p],\mathbb{Z})_\blacksquare) \to D(\mathbb{P}^{1,\mathrm{ad}}_{\mathbb{Q}_p}) \to D(\mathbb{Q}_p\langle T\rangle_\blacksquare).$$

We are now ready to construction cohomology with compact support for the affine line.

Theorem 8.13. *cohomology with compact support*

9. Solid Quasicoherent Sheaves

10. Solid Quasicoherent Six-Functors

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